

CMV BIORTHOGONAL LAURENT POLYNOMIALS. II: CHRISTOFFEL FORMULAS FOR GERONIMUS–UVAROV TRANSFORMATIONS

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ABSTRACT. This paper is a continuation of the recent paper *CMV biorthogonal Laurent polynomials: Christoffel formulas for Christoffel and Geronimus transformations* by the same authors. The behavior of quasidefinite sesquilinear forms for Laurent polynomials in the complex plane, characterized by bivariate linear functionals, and corresponding CMV biorthogonal Laurent polynomial families—including Sobolev and discrete Sobolev orthogonalities—under two type of Geronimus–Uvarov transformations is studied. Either the linear functionals are multiplied by a Laurent polynomial and divided by the complex conjugate of a Laurent polynomial, with the addition of appropriate masses (linear functionals supported on the zeros of the perturbing Laurent polynomial in the denominator) or vice-versa, multiplied by the complex conjugate of a Laurent polynomial and divided by a Laurent polynomial. The connection formulas for the CMV biorthogonal Laurent polynomials, their norms, and Christoffel–Darboux kernels are given. For prepared Laurent polynomials, i.e. of the form $L_{\mathcal{C}}(z) = L_{\mathcal{C},N_{\mathcal{C}}}z^{N_{\mathcal{C}}} + \cdots + L_{\mathcal{C},-N_{\mathcal{C}}}z^{-N_{\mathcal{C}}}$, $L_{\mathcal{C},N_{\mathcal{C}}}L_{\mathcal{C},-N_{\mathcal{C}}} \neq 0$ and $L_{\Gamma}(z) = L_{\Gamma,N_{\Gamma}}z^{N_{\Gamma}} + \cdots + L_{\Gamma,-N_{\Gamma}}z^{-N_{\Gamma}}$, $L_{\Gamma,N_{\Gamma}}L_{\Gamma,-N_{\Gamma}} \neq 0$, these connection formulas lead to quasideterminantal (quotient of determinants) Christoffel formulas expressing an arbitrary degree perturbed biorthogonal Laurent polynomial in terms of $2N_{\mathcal{C}} + 2N_{\Gamma}$ unperturbed biorthogonal Laurent polynomials, their second kind functions or Christoffel–Darboux kernel and its mixed versions. When the linear functionals are supported on the unit circle, a particularly relevant role is played by the reciprocal polynomial, and the Christoffel formulas provide now with two possible ways of expressing the same perturbed quantities in terms of the original ones, one using only the nonperturbed biorthogonal family of Laurent polynomials, and the other using the Christoffel–Darboux kernel and its mixed versions.

1. INTRODUCTION

In this paper we continue with the discussion of [12] regarding transformations of CMV biorthogonal Laurent polynomials. In that precedent paper we studied Christoffel and Geronimus perturbations and the corresponding Christoffel formulas and in this one we complete the analysis with the Geronimus–Uvarov perturbations, which could be thought as a composition of a Christoffel and a Geronimus transformation. For a deeper historical description and a more extended discussion of the state of the art regarding these issues we refer to the previous paper [12]. Here we just reproduce some of the more essential facts for the discussion of the Geronimus–Uvarov transformations.

Christoffel formulas [20] for perturbations, $\hat{u} = p(x)u$, where u is a linear functional and $p(x)$ is a polynomial, constitutes a classical result in the theory of orthogonal polynomials [19, 41, 23]. Connection formulas between two families of orthogonal polynomials allow to express any polynomial of a given degree n as a linear combination of all polynomials of degree less than or equal to n in the second family. Remarkably, for the Christoffel formulas, which are connection formulas, the number of terms does not grow with the degree n but remain constant, equal to the degree of the perturbing polynomial. The problem

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of given two functionals \tilde{u} and u such that $\tilde{u}_x = \frac{p(x)}{q(x)}u_x$ where $p(x), q(x)$ are polynomials was analyzed by Uvarov in [43] when u, v are positive definite measures supported on the real line. See also [45] for a discussion including spectral masses, in where the perturbation is written in the $q(x)\tilde{u}_x = p(x)u_x$, in that paper Zhedanov named these transformations as linear spectral transformations. We prefer to call them Geronimus–Uvarov.

Orthogonal polynomials in the unit circle \mathbb{T} or Szegő polynomials are monic polynomials P_n of degree n such that $\int_{\mathbb{T}} P_n(z)z^{-k} d\mu(z) = 0$, for $k = 0, 1, \dots, n-1$ [41]. The extension to this context of the three-term relations and tridiagonal Jacobi matrices needs of Hessenberg matrices and give the Szegő recursion relation, which is expressed in terms of the reciprocal Szegő polynomials and the Verblunsky coefficients. The papers [31, 32] on the strong Stieltjes moment problem can be considered as the starting point for the consideration of orthogonal Laurent polynomials on the real line. For recursion relations, Favard’s theorem, quadrature problems, and Christoffel–Darboux formulas for Laurent polynomials on the unit circle \mathbb{T} see [42, 15, 17, 13, 14]. Orthogonal Laurent polynomials are dense in $L^2(\mathbb{T}, \mu)$, but Szegő polynomials are not in general [16] and [15]. The CMV (Cantero–Moral–Velázquez) matrices [17] constitute a representation of the multiplication operator in terms of the basis of orthonormal Laurent polynomials and where discussed in [18] in connection with Darboux transformations. In [27] extensions of the Christoffel determinantal type formulas were given for the analogue of the Christoffel transformation, with an arbitrary degree polynomial having multiple roots, using the original Szegő polynomials and its Christoffel–Darboux kernels. The Geronimus transformation for OPUC, with a perturbation of degree 2 and no masses, was discussed in [28]. In [30], alternative formulas *à la Christoffel*, not based on the Christoffel–Darboux kernel [27], were given in terms of determinantal expressions of the Szegő polynomials and their reverse polynomials, also as Uvarov did in [43], they considered multiplication by rational functions, but no masses at all where discussed in this paper. In [38] some concrete cases where considered within the biorthogonal scenario. The transformations considered in this work are also known as Darboux transformations [33]. Indeed, in the context of the Sturm–Liouville theory, Darboux discussed in [21] a dimensional simplification of a geometrical transformation in two dimensions founded previously [34] which can be considered, as we called it today, a *Darboux transformation*.

Our discussion framework is constructed upon the *noyaux-distribution* [39]. A space of fundamental functions, in the sense of [24, 25], and the corresponding space of generalized functions provides with a linear functional setting for orthogonal polynomials. Discrete orthogonality appears when we consider linear functionals with discrete and infinite support [35]. We will consider an arbitrary nondegenerate continuous sesquilinear form given by a generalized kernel u_{z_1, \bar{z}_2} with a quasidefinite Gram matrix. This scheme not only contains the more usual choices of Gram matrices like those of Toeplitz type on the unit circle, or those leading to discrete orthogonality but also Sobolev orthogonality.

The Gauss–Borel factorization problem, has been applied by our group in Madrid not only to the Christoffel and Geronimus transformations for sequilinear forms [12] but also to the following cases

- i) Laurent orthogonal polynomials on the unit circle in [4].
- ii) Some extensions of the Christoffel–Darboux formula to generalized orthogonal polynomials [1] and to multiple orthogonal polynomials, [5, 8].
- iii) Christoffel transformations and the relation with non-Abelian Toda hierarchies for real matrix orthogonal polynomials were studied in [2], and in [3] we extended those results to include the Geronimus, Geronimus–Uvarov and Uvarov transformations.
- iv) Multiple orthogonal polynomials and multicomponent Toda [6].
- v) For matrix orthogonal Laurent polynomials on the unit circle, CMV orderings, and non-Abelian lattices on the circle [7].
- vi) Multivariate orthogonal polynomials in several real variables and corresponding multispectral integrable Toda hierarchy [9, 10]. Multivariate orthogonal polynomials on the multidimensional unit torus, the multivariate extension of the CMV ordering and integrable Toda hierarchies [11].

1.1. Objectives, results, layout of the paper and perspectives. In the paper we continue and extend the studies of [12]. As in that paper, we consider a general sesquilinear form in the complex plane determined by a bivariate linear functional, its biorthogonal Laurent families and their behavior under Geronimus–Uvarov perturbations, that can be thought as an appropriate consecutive composition of a Geronimus and a Christoffel perturbation. Both of these transformations were analyzed in [12].

The ideas of [12] are used again in this paper, namely we consider a Gauss–Borel factorization of the Gram matrix, which we assume to be quasidefinite, this will lead to connection formulas for the biorthogonal Laurent polynomial families, the corresponding second kind functions and the standard and mixed Christoffel–Darboux kernels. We find determinantal Christoffel formulas for the Geronimus–Uvarov transformation. Let us stress the relation of the results of the present paper and the previous works [27, 28, 30]

- i) In [27, 28, 30] the sesquilinear forms are supported on the diagonal with linear functionals of zero order and positive (given, therefore, by a positive Borel measure). Moreover, the studies in [27, 30] are restricted to measures supported on the unit circle. Our scheme allows for a more general biorthogonality and therefore includes Sobolev orthogonality and discrete Sobolev orthogonality with arbitrary support (with an infinity number points) on the complex plane.
- ii) The papers [27, 28] do not consider Geronimus–Uvarov transformations. In [27] only the Christoffel transformations for orthogonal polynomials on the unit circle is analyzed, and in [28] a particular Geronimus transformation of degree two, with no masses, is discussed. Regarding Geronimus–Uvarov transformations in the unit circle, [30] does not incorporate masses at all. In our paper we include a very general class of masses.

We have studied two possible Geronimus–Uvarov transformations, and give the corresponding Christoffel formulas. These two transformations can be made to coincide when we have an initial zero order diagonal case supported on the unit circle, the Toeplitz situation. Then, analogously as what we discovered in [12], two alternative Christoffel formulas emerge.

The layout of the paper is as follows. We now proceed with a resume regarding basic facts about CMV biorthogonal Laurent polynomials. In §2 we perturb a general quasidefinite sesquilinear form by multiplying the corresponding bivariate linear functional by a quotient of a Laurent polynomial and the complex conjugate of another Laurent polynomial, one depending on the first variable of the bivariate linear functional and the other in the second variable, and another by the complex conjugate of the previous quotient. We include the addition of masses supported on the zeros of the Laurent polynomials in the denominator of the perturbation. Quasidefiniteness of the perturbed sesquilinear forms allows for the Gauss–Borel factorization, which leads to connection formulas for the biorthogonal Laurent polynomials, second kind functions, Christoffel–Darboux kernels, and mixed Christoffel–Darboux kernels, see Propositions 8, 10, 12, and 13. With this at hand we present Theorem 1, where the Christoffel–Geronimus–Uvarov formulas for both type of perturbations are given. We write, for the first time, this type of expressions including the masses. In §3 we discuss possible reductions, first to zero order supported on the diagonal, then to the unit circle, and finally linear functionals taking real values. There is an Appendix containing some of the proofs.

1.2. Basic facts regarding CMV biorthogonal Laurent polynomials.

Definition 1 (Sesquilinear forms). *A sesquilinear form $\langle \cdot, \cdot \rangle$ in complex linear space V is a continuous map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ such that for any triple $f, g, h \in V$ the following conditions are satisfied*

- i) $\langle Af + Bg, h \rangle = A \langle f, h \rangle + B \langle g, h \rangle, \forall A, B \in \mathbb{C},$
- ii) $\langle f, Ag + Bh \rangle = \langle f, g \rangle \bar{A} + \langle f, h \rangle \bar{B}, \forall A, B \in \mathbb{C}.$

Given the ordered bi-infinite basis $\{z^l\}_{l=-\infty}^{\infty}$ or the semiinfinite CMV basis $\{\chi^{(l)}(z)\}_{l=0}^{\infty}$, where we have $\chi^{(l)} := \begin{cases} z^{l/2}, & l \text{ even} \\ z^{-(l+1)/2}, & l \text{ odd.} \end{cases}$ of $\mathbb{C}[z, z^{-1}]$ the sesquilinear form is characterized by the corresponding Gram

matrix. For example, in the first case we have the biinfinite Gram matrix $g = \begin{bmatrix} g_{0,0} & g_{0,1} & \dots \\ g_{1,0} & g_{1,1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$ with $g_{k,l} = \langle (z_1)^k, (z_2)^l \rangle, k, l \in \mathbb{Z}$.

Definition 2 (Laurent polynomial spectrum). *The zero set of a function $L(z)$ in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ will be denoted by $\sigma(L)$ and said to be the spectrum of $L(z)$.*

In this paper we will consider sesquilinear forms constructed in terms of bivariate linear functionals with well-defined support. We will work with generalized functions \mathcal{F}' , i.e. continuous linear functionals over the space of fundamental functions \mathcal{F} , [24, 25], such that $\mathbb{C}[z, z^{-1}] \subsetneq \mathcal{F}$ and there is a well defined notion of support (this last condition forbids the choice $\mathbb{C}[z, z^{-1}] = \mathcal{F}$). The space of distributions is the space of generalized functions when the fundamental functions space is the set of complex smooth functions [40] with compact support $\mathcal{D}^* := C_0^\infty(\mathbb{C}^*)$. The zero set of a distribution $u \in (\mathcal{D}^*)'$ is the open region $\Omega \subset \mathbb{C}^*$ whenever for every $f(z)$ supported on Ω we have $\langle u, f \rangle = 0$. Its complementary set, which is closed, is the support, $\text{supp}(u)$, of the distribution u . Obviously $\mathbb{C}[z, z^{-1}] \not\subset \mathcal{D}$ and, consequently, the space of test functions \mathcal{D} is not suitable for our aims. However, the next example gives an adequate scenario for our constructions. When we take as our space of fundamental functions \mathcal{F} the space of smooth functions in \mathbb{C}^* , $\mathcal{F} = \mathcal{E}^* = C^\infty(\mathbb{C}^*)$ the corresponding space of generalized functions is the space $(\mathcal{E}^*)'$ of distributions of compact support in \mathbb{C}^* . Now, as $\mathbb{C}[z, z^{-1}] \subsetneq \mathcal{E}^*$ we deduce that $(\mathcal{E}^*)' \subsetneq (\mathbb{C}[z, z^{-1}])' \cap (\mathcal{D}^*)'$. The set of distributions of compact support is a first example of an appropriate framework for the consideration of polynomials and supports simultaneously. For any bivariate linear functional u_{z_1, \bar{z}_2} with support $\text{supp}(u_{z_1, \bar{z}_2})$ its projections in the axis z_i are denoted by $\text{supp}_i(u_{z_1, \bar{z}_2})$, $i = 1, 2$. Sesquilinear forms are constructed in terms of bivariate linear functionals.

Definition 3. *We consider the following sesquilinear forms*

$$\langle f(z_1), g(z_2) \rangle_u = \left\langle u_{z_1, \bar{z}_2}, f(z_1) \otimes \overline{g(z_2)} \right\rangle, \quad f(z), g(z) \in \mathcal{F}.$$

Hence, the following sesquilinear forms $\langle f(z_1), g(z_2) \rangle_u = \sum_{0 \leq n, m < \infty} \int \frac{\partial^n f}{\partial z^n}(z_1) \overline{\frac{\partial^m g}{\partial z^m}(z_2)} d\mu^{(m,n)}(z_1, z_2)$, for Borel measures $\mu^{(m,n)}(z_1, z_2)$ in \mathbb{C}^2 , with at least one of them with infinite support, are included in our considerations. In the bi-infinite basis $\{z^n\}_{n \in \mathbb{Z}}$ we have the Gram matrix $g = [g_{n,m}]$ with $g_{n,m} = \langle (z_1)^n, (z_2)^m \rangle_u = \langle u_{z_1, \bar{z}_2}, z_1 \otimes (\bar{z}_2)^m \rangle$.

Following [17, 44], we will use the CMV basis $\{\chi^{(0)}, \chi^{(1)}, \chi^{(2)}, \dots\}$ with $\chi^{(l)}(z) = \begin{cases} z^k, & l = 2k, \\ z^{-k-1}, & l = 2k+1. \end{cases}$

Definition 4. *Let us consider*

$$\chi_1(z) := [1, 0, z, 0, z^2, 0, \dots]^\top, \quad \chi_2(z) := [0, 1, 0, z, 0, z^2, 0, \dots]^\top$$

and

$$\chi_1^*(z) := z^{-1}\chi_1(z^{-1}) = [z^{-1}, 0, z^{-2}, 0, z^{-3}, 0, \dots]^\top, \quad \chi_2^*(z) := z^{-1}\chi_2(z^{-1}) = [0, z^{-1}, 0, z^{-2}, 0, z^{-3}, 0, \dots]^\top.$$

In terms of which we define the CMV sequences

$$\chi(z) := \chi_1(z) + \chi_2^*(z) = [1, z^{-1}, z, z^{-2}, \dots]^\top, \quad \chi^*(z) := \chi_1^*(z) + \chi_2(z) = [z^{-1}, 1, z^{-2}, z, z^{-3}, z^2, \dots]^\top.$$

The matrices

$$\Upsilon = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Given a bivariate linear functional $\mathbf{u}_{z_1, \bar{z}_2}$ and the associated sesquilinear form we consider the corresponding Gram matrix

$$G = \left\langle \chi(z_1), (\chi(z_2))^\top \right\rangle_{\mathbf{u}} = \left\langle \mathbf{u}_{z_1, \bar{z}_2}, \chi(z_1) \otimes (\chi(z_2))^\dagger \right\rangle.$$

Proposition 1. The semi-infinite matrix Υ , is unitary $\Upsilon^\top = \Upsilon^{-1}$, and has the important spectral properties $\Upsilon\chi(z) = z\chi(z)$ and $\Upsilon^{-1}\chi(z) = z^{-1}\chi(z)$.

For the truncation of semi-infinity matrices we will use the following notation $A = \begin{bmatrix} A_{0,0} & A_{0,1} & \dots \\ A_{1,0} & A_{1,1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and $A^{[l]} := \begin{bmatrix} A_{0,0} & A_{0,1} & \dots & A_{0,l-1} \\ A_{1,0} & A_{1,1} & \dots & A_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l-1,0} & A_{l-1,1} & \dots & A_{l-1,l-1} \end{bmatrix}$. For the corresponding block structure we write $A = \left[\begin{array}{c|c} A^{[l]} & A^{[l, \geq l]} \\ \hline A^{[\geq l, l]} & A^{[\geq l]} \end{array} \right]$.

We will assume that the Gram matrix G is quasidefinite, i.e., all its principal minors are not zero, so that the following Gauss–Borel or LU factorization of G holds

$$(1) \quad G = S_1^{-1} H (S_2^{-1})^\dagger,$$

where S_1 and S_2 are lower unitriangular matrices and H is a diagonal nonsingular matrix.

Definition 5. Let us introduce the following vectors of Laurent polynomials

$$(2) \quad \phi_1(z) := S_1 \chi(z), \quad \phi_2(z) := S_2 \chi(z).$$

Its components $\phi_1(z) = [\phi_{1,0}(z), \phi_{1,1}(z), \dots]^\top$ and $\phi_2(z) = [\phi_{2,0}(z), \phi_{2,1}(z), \dots]^\top$ are such that

Proposition 2 (Biorthogonal polynomials). The following biorthogonality conditions

$$\langle \phi_{1,n}(z_1), \phi_{2,m}(z_2) \rangle_{\mathbf{u}} = \delta_{n,m} H_n,$$

hold for $n, m \in \{0, 1, 2, \dots\}$.

Corollary 1. The orthogonality relations

$$\begin{aligned} \langle \phi_{1,2k}(z_1), (z_2)^l \rangle_{\mathbf{u}} &= 0, & -k \leq l \leq k-1, \\ \langle \phi_{1,2k+1}(z_1), (z_2)^l \rangle_{\mathbf{u}} &= 0, & -k \leq l \leq k, \\ \langle (z_1)^l, \phi_{2,2k}(z_2) \rangle_{\mathbf{u}} &= 0, & -k \leq l \leq k-1, \\ \langle (z_1)^l, \phi_{2,2k+1}(z_2) \rangle_{\mathbf{u}} &= 0, & -k \leq l \leq k, \end{aligned}$$

are satisfied.

Definition 6. *The Christoffel–Darboux kernel is*

$$(3) \quad K^{[l]}(\bar{z}_1, z_2) := \sum_{k=0}^{l-1} \overline{\phi_{2,k}(z_1)} H_k^{-1} \phi_{1,k}(z_2) = [\phi_2(z_1)^\dagger]^{[l]} (H^{-1})^{[l]} [\phi_1(z_2)]^{[l]}.$$

Proposition 3. *The Christoffel–Darboux kernel satisfies the projection properties*

$$\left\langle \sum_{j=0}^M f_j \phi_{1,j}(z_1), \overline{K^{[l]}(\bar{z}_2, z)} \right\rangle_u = \sum_{j=0}^{l-1} f_j \phi_{1,j}(z), \quad \left\langle K^{[l]}(\bar{z}, z_1), \sum_{j=0}^M f_j \phi_{2,j}(z_2) \right\rangle_u = \sum_{j=0}^{l-1} \overline{f_j \phi_{2,j}(z)}.$$

This implies that, when acting on the right $\overline{K^{[l+1]}(\bar{z}_2, z)}$ projects over $\Lambda_l := \mathbb{C}\{\chi^{(k)}(z)\}_{k=0}^l$ while when acting on the left $K^{[l+1]}(\bar{z}, z_1)$ projects on $\overline{\Lambda_l}$. Notice that

$$\Lambda_{2k} = \{1, z^{-1}, z, \dots, z^{-k}, z^k\}, \quad \Lambda_{2k+1} = \{1, z^{-1}, z, \dots, z^k, z^{-k-1}\}.$$

Corollary 2. *If $L(z) \in \Lambda_l := \mathbb{C}\{\chi^{(k)}(z)\}_{k=0}^{l-1}$ then*

$$\left\langle L(z_1), \overline{K^{[l]}(\bar{z}_2, z)} \right\rangle_u = L(z), \quad \left\langle K^{[l]}(\bar{z}, z_1), L(z_2) \right\rangle_u = \overline{L(z)}.$$

Definition 7. *The second kind functions are given by*

$$C_1(z) = \left\langle \phi_1(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_u = \left\langle u_{z_1, \bar{z}_2}, \phi_1(z_1) \otimes \frac{1}{z - \bar{z}_2} \right\rangle, \quad z \notin \overline{\text{supp}_2(u)},$$

$$(C_2(z))^\dagger = \left\langle \frac{1}{\bar{z} - z_1}, (\phi_2(z_2))^\top \right\rangle_u = \left\langle u_{z_1, \bar{z}_2}, \frac{1}{\bar{z} - z_1} \otimes (\phi_2(z_2))^\dagger \right\rangle, \quad z \notin \overline{\text{supp}_1(u)}.$$

Definition 8. *The mixed Christoffel–Darboux kernels are*

$$(4) \quad K_{C_2}^{[l]}(\bar{z}_1, z_2) := \sum_{k=0}^{l-1} \overline{C_{2,k}(z_1)} H_k^{-1} \phi_{1,k}(z_2) = [C_2(z_1)^\dagger]^{[l]} (H^{-1})^{[l]} [\phi_1(z_2)]^{[l]}, \quad z_1 \notin \overline{\text{supp}_1(u)},$$

$$(5) \quad K_{C_1}^{[l]}(\bar{z}_1, z_2) := \sum_{k=0}^{l-1} \overline{\phi_{2,k}(z_1)} H_k^{-1} C_{1,k}(z_2) = [\phi_2(z_1)^\dagger]^{[l]} (H^{-1})^{[l]} [C_1(z_2)]^{[l]}, \quad z_2 \notin \overline{\text{supp}_2(u)}.$$

Proposition 4. *The mixed kernels have the following expressions*

$$K_{C_2}^{[l]}(\bar{x}_1, x_2) = \left\langle \frac{1}{\bar{x}_1 - z_1}, \overline{K^{[l]}(\bar{z}_2, x_2)} \right\rangle_u, \quad K_{C_1}^{[l]}(\bar{x}_1, x_2) := \left\langle K^{[l]}(\bar{x}_1, z_1), \frac{1}{\bar{x}_2 - z_2} \right\rangle_u.$$

Hence, the mixed kernels can be thought as the projections of the Cauchy kernels or, equivalently, the Cauchy transforms of the Christoffel–Darboux kernels.

Definition 9 (Prepared Laurent polynomials). *For every $2n$ -degree polynomial $P(z) = P_{2n}z^{2n} + \dots + P_0 \in \mathbb{C}[z]$ with $P_0 \neq 0$, its Féjer–Riesz corresponding Laurent polynomial is given by*

$$(6) \quad L(z) = z^{-n}P(z) = L_n z^n + \dots + L_{-n} z^{-n}, \quad L_n = P_{2n}, \quad L_{-n} = P_0.$$

We say that a Laurent polynomial is prepared whenever it is the Féjer–Riesz corresponding Laurent polynomial of an even degree polynomial non vanishing at the origin.

For the consideration of arbitrary multiplicities of the zeros of the perturbing polynomials we need of

Definition 10 (Spectral jets). *Given a Laurent polynomial $L(z)$ with zeros and multiplicities $\{\zeta_i, m_i\}_{i=1}^d$ we introduce the spectral jet of a function $f(z)$ along $L(z)$ as follows:*

$$\mathcal{J}_f^L := \left[f(\zeta_1), f'(\zeta_1), \dots, \frac{f^{(m_1-1)}(\zeta_1)}{(m_1-1)!}, \dots, f(\zeta_d), f'(\zeta_d), \dots, \frac{f^{(m_d-1)}(\zeta_d)}{(m_d-1)!} \right] \in \mathbb{C}^{2m}.$$

2. GERONIMUS–UVAROV TRANSFORMATIONS

We now proceed with the consideration of the Geronimus–Uvarov transformation in the context of complex sesquilinear forms. We will present the extension of the formulas found by Uvarov [43] for this situation, but now including masses. We continue with the discussion of [12] and compose a first Geronimus transformation with a consecutive Christoffel transformation, in that precise order, and as we have two types of such perturbations, either if we multiply or divide by a Laurent polynomial or the complex conjugate of a Laurent polynomial, we have two type of Geronimus–Uvarov perturbations, namely

$$\tilde{u}_{z_1, \bar{z}_2}^{(1,2)} = \overline{(\tilde{u}_{z_1, \bar{z}_2}^{(1)})}^{(2)} \quad \text{and} \quad \tilde{u}_{z_1, \bar{z}_2}^{(2,1)} = \overline{(\tilde{u}_{z_1, \bar{z}_2}^{(2)})}^{(1)}. \quad \text{Let us now proceed with the precise definition.}$$

Definition 11. For a bivariate linear functional u_{z_1, \bar{z}_2} with a well-defined support, for $a \in \{1, 2\}$, given Laurent polynomials $L_\Gamma^{(a)}(z) = L_{\Gamma, N_\Gamma^+}^{(a)} z^{N_\Gamma^+} + \cdots + L_{\Gamma, -N_\Gamma^-}^{(a)} z^{-N_\Gamma^-}$, and $L_\mathcal{C}^{(a)}(z) = L_{\mathcal{C}, N_\mathcal{C}^+}^{(a)} z^{N_\mathcal{C}^+} + \cdots + L_{\mathcal{C}, -N_\mathcal{C}^-}^{(a)} z^{-N_\mathcal{C}^-}$ such that $L_{\Gamma, N_\Gamma^+}^{(a)} L_{\Gamma, -N_\Gamma^-}^{(a)} \neq 0$, $L_{\mathcal{C}, N_\mathcal{C}^+}^{(a)} L_{\mathcal{C}, -N_\mathcal{C}^-}^{(a)} \neq 0$, $N_\Gamma^\pm, N_\mathcal{C}^\pm \in \{1, 2, \dots\}$, and $\sigma(L_\Gamma^{(1)}(z)) \cap \text{supp}_1 u = \emptyset$, $\sigma(L_\Gamma^{(2)}(z)) \cap \text{supp}_2 u = \emptyset$, we consider two possible families of Geronimus–Uvarov transformations $\tilde{u}_{z_1, \bar{z}_2}^{(1,2)}$ and $\tilde{u}_{z_1, \bar{z}_2}^{(2,1)}$ characterized by

$$L_\Gamma^{(1)}(z_1) \tilde{u}_{z_1, \bar{z}_2}^{(1,2)} = u_{z_1, \bar{z}_2} \overline{L_\mathcal{C}^{(2)}(z_2)}, \quad \tilde{u}_{z_1, \bar{z}_2}^{(2,1)} \overline{L_\Gamma^{(2)}(z_2)} = L_\mathcal{C}^{(1)}(z_1) u_{z_1, \bar{z}_2}.$$

The notation $L_\Gamma^{(a)}(z)$ makes reference to a perturbation of Geronimus type while $L_\mathcal{C}^{(a)}(z)$ to a perturbation of Christoffel type. Therefore, the perturbed bivariate linear functionals are

$$\begin{aligned} \tilde{u}_{z_1, \bar{z}_2}^{(1,2)} &= \frac{\overline{L_\mathcal{C}^{(2)}(z_2)}}{L_\Gamma^{(1)}(z_1)} u_{z_1, \bar{z}_2} + \overline{L_\mathcal{C}^{(2)}(z_2)} \sum_{i=1}^{d^{(1)}} \sum_{l=0}^{m_i^{(1)}-1} \frac{(-1)^l}{l!} \delta^{(l)}(z_1 - \zeta_i^{(1)}) \otimes \overline{(\xi_{i,l}^{(1)})_{z_2}}, \\ \tilde{u}_{z_1, \bar{z}_2}^{(2,1)} &= \frac{L_\mathcal{C}^{(1)}(z_1)}{\overline{L_\Gamma^{(2)}(z_2)}} u_{z_1, \bar{z}_2} + L_\mathcal{C}^{(1)}(z_1) \sum_{i=1}^{d^{(2)}} \sum_{l=0}^{m_i^{(2)}-1} \frac{(-1)^l}{l!} (\xi_{i,l}^{(2)})_{z_1} \otimes \overline{\delta^{(l)}(z_2 - \zeta_i^{(2)})}, \end{aligned}$$

where $\zeta_i^{(a)}$ are zeros with multiplicities $m_i^{(a)}$ of $L_\Gamma^{(a)}(z_1)$, while $(\xi_{i,l}^{(a)})_{z_a}$ are univariate linear functionals. In terms of sesquilinear forms we have

$$\left\langle L_\Gamma^{(1)}(z_1) f(z_1), g(z_2) \right\rangle_{\tilde{u}^{(1,2)}} = \left\langle f(z_1), L_\mathcal{C}^{(2)}(z_2) g(z_2) \right\rangle_u, \quad \left\langle f(z_1), L_\Gamma^{(2)}(z_2) g(z_2) \right\rangle_{\tilde{u}^{(2,1)}} = \left\langle L_\mathcal{C}^{(1)}(z_1) f(z_1), g(z_2) \right\rangle_u,$$

for all $f, g \in \mathcal{F}$.

Proposition 5. Geronimus–Uvarov transformations associated with the two couples of perturbing Laurent polynomials $L_\Gamma^{(1)}(z), L_\mathcal{C}^{(2)}(z)$ and $L_\Gamma^{(2)}(z), L_\mathcal{C}^{(1)}(z)$ imply for the corresponding Gram matrices

$$(7) \quad L_\Gamma^{(1)}(\Upsilon) \tilde{G}^{(1)} = G(L_\mathcal{C}^{(2)}(\Upsilon))^\dagger, \quad \tilde{G}^{(2)}(L_\Gamma^{(2)}(\Upsilon))^\dagger = L_\mathcal{C}^{(1)}(\Upsilon) G.$$

Proof. It follows from

$$\begin{aligned} L_\Gamma^{(1)}(\Upsilon) \tilde{G}^{(1)} &= \left\langle \tilde{u}_{z_1, \bar{z}_2}^{(1,2)}, L_\Gamma^{(1)}(z_1) \chi(z_1) \otimes (\chi(z_2))^\dagger \right\rangle \\ &= \left\langle u_{z_1, \bar{z}_2}, \chi(z_1) \otimes (\chi(z_2))^\dagger \overline{L_\mathcal{C}^{(2)}(z_2)} \right\rangle \\ &= \left\langle u_{z_1, \bar{z}_2}, \chi(z_1) \otimes (\chi(z_2))^\dagger \right\rangle (L_\mathcal{C}^{(2)}(\Upsilon))^\dagger, \\ \tilde{G}^{(2)}(L_\Gamma^{(2)}(\Upsilon))^\dagger &= \left\langle \tilde{u}_{z_1, \bar{z}_2}^{(2,1)}, \chi(z_1) \otimes (\chi(z_2))^\dagger \overline{L_\Gamma^{(2)}(z_2)} \right\rangle \\ &= \left\langle u_{z_1, \bar{z}_2}, L_\mathcal{C}^{(1)}(z_1) \chi(z_1) \otimes (\chi(z_2))^\dagger \right\rangle \\ &= L_\mathcal{C}^{(1)}(\Upsilon) \left\langle u_{z_1, \bar{z}_2}, \chi(z_1) \otimes (\chi(z_2))^\dagger \right\rangle. \end{aligned}$$

□

We assume that both Gram matrices are quasidefinite, i.e. the Gauss–Borel factorizations

$$(8) \quad \tilde{G}^{(1)} = (\tilde{S}_1^{(1)})^{-1} \tilde{H}^{(1,2)} (\tilde{S}_2^{(1)})^{-\dagger}, \quad \check{G}^{(2)} = (\check{S}_1^{(2)})^{-1} \check{H}^{(2,1)} (\check{S}_2^{(2)})^{-\dagger},$$

can be performed.

2.1. Connection formulas for the CMV biorthogonal Laurent polynomials and its second kind functions.

Definition 12 (Geronimus–Uvarov connectors). *For a bivariate linear functional and Geronimus–Uvarov transformations and two couples of perturbing Laurent polynomials $\{L_\Gamma^{(1)}(z), L_\mathfrak{C}^{(1)}(z)\}$ and $\{L_\Gamma^{(2)}(z), L_\mathfrak{C}^{(2)}(z)\}$, we associate the following connectors*

$$\begin{aligned} \Omega_1^{(1,2)} &= S_1 L_\Gamma^{(1)}(\gamma) (\tilde{S}_1^{(1)})^{-1}, & \Omega_2^{(1,2)} &= \tilde{S}_2^{(1)} L_\mathfrak{C}^{(2)}(\gamma) (S_2)^{-1}, \\ \Omega_1^{(2,1)} &= \tilde{S}_1^{(2)} L_\mathfrak{C}^{(1)}(\gamma) (S_1)^{-1}, & \Omega_2^{(2,1)} &= S_2 L_\Gamma^{(2)}(\gamma) (\check{S}_2^{(2)})^{-1}. \end{aligned}$$

Proposition 6. *Geronimus–Uvarov connectors satisfy*

$$\Omega_1^{(1,2)} \tilde{H}^{(1,2)} = H(\Omega_2^{(1,2)})^\dagger, \quad \check{H}^{(2,1)} (\Omega_2^{(2,1)})^\dagger = \Omega_1^{(2,1)} H.$$

Proof. From (7) and (8) we deduce that

$$\begin{aligned} L_\Gamma^{(1)}(\gamma) (\tilde{S}_1^{(1)})^{-1} \tilde{H}^{(1,2)} (\tilde{S}_2^{(1)})^{-\dagger} &= (S_1)^{-1} H (S_2)^{-\dagger} (L_\mathfrak{C}^{(2)}(\gamma))^\dagger, \\ (\check{S}_1^{(2)})^{-1} \check{H}^{(2,1)} (\check{S}_2^{(2)})^{-\dagger} (L_\Gamma^{(2)}(\gamma))^\dagger &= L_\mathfrak{C}^{(1)}(\gamma) (S_1)^{-1} H (S_2)^{-\dagger}. \end{aligned}$$

Hence

$$S_1 L_\Gamma^{(1)}(\gamma) (\tilde{S}_1^{(1)})^{-1} \tilde{H}^{(1,2)} = H (S_2)^{-\dagger} (L_\mathfrak{C}^{(2)}(\gamma))^\dagger (\tilde{S}_2^{(1)})^\dagger, \quad \check{H}^{(2,1)} (\check{S}_2^{(2)})^{-\dagger} (L_\Gamma^{(2)}(\gamma))^\dagger (S_2)^\dagger = \tilde{S}_1^{(2)} L_\mathfrak{C}^{(1)}(\gamma) (S_1)^{-1} H.$$

□

Definition 13. *For our convenience we introduce for $\rho = \mathfrak{C}, \Gamma$*

$$\begin{aligned} L_{\rho, -\underline{N}_\rho}^{(\mathfrak{a})} &:= 0, & \text{for } -\underline{N}_\rho < -N_\rho^-, \\ L_{\rho, \underline{N}_\rho}^{(\mathfrak{a})} &:= 0, & \text{for } \underline{N}_\rho > N_\rho^+. \end{aligned}$$

Proposition 7. *Geronimus–Uvarov connectors are banded semi-infinite matrices. In particular, if $\underline{N}_\Gamma := \max(N_\Gamma^+, N_\Gamma^-)$ and $\underline{N}_\mathfrak{C} := \max(N_\mathfrak{C}^+, N_\mathfrak{C}^-)$*

- i) *The connectors $\Omega_1^{(1,2)}$ and $\Omega_2^{(2,1)}$ have as possible nonzero diagonals the first $2\underline{N}_\Gamma$ superdiagonals and $2\underline{N}_\mathfrak{C}$ subdiagonals.*
- ii) *The connectors $\Omega_2^{(1,2)}$ and $\Omega_1^{(2,1)}$ have as possible nonzero diagonals the first $2\underline{N}_\mathfrak{C}$ superdiagonals and $2\underline{N}_\Gamma$ subdiagonals.*
- iii) *Moreover, we have the formulas*

$$(9) \quad (\Omega_2^{(1,2)})_{l, l-2\underline{N}_\Gamma} = \overline{L_{\Gamma, (-1)^l \underline{N}_\Gamma}^{(1)} \frac{\tilde{H}_l^{(1,2)}}{H_{k-2\underline{N}_\Gamma}}}, \quad (\Omega_1^{(2,1)})_{l, l-2\underline{N}_\Gamma} = \overline{L_{\Gamma, (-1)^l \underline{N}_\Gamma}^{(2)} \frac{\tilde{H}_l^{(2,1)}}{H_{l-2\underline{N}_\Gamma}}}, \quad l \geq 2\underline{N}_\Gamma,$$

$$(10) \quad (\Omega_2^{(1,2)})_{l, l+2\underline{N}_\mathfrak{C}} = L_{\mathfrak{C}, (-1)^l \underline{N}_\mathfrak{C}}^{(2)}, \quad (\Omega_1^{(2,1)})_{l, l+2\underline{N}_\mathfrak{C}} = L_{\mathfrak{C}, (-1)^l \underline{N}_\mathfrak{C}}^{(1)}.$$

To name these semi-infinite matrices as connectors is justified by

Proposition 8. *The following connection formulas for the CMV biorthogonal Laurent polynomials hold*

$$\begin{aligned} \Omega_1^{(1,2)} \tilde{\phi}_1^{(1,2)}(z) &= L_\Gamma^{(1)}(z) \phi_1(z), & \Omega_2^{(1,2)} \phi_2(z) &= L_\mathfrak{C}^{(2)}(z) \tilde{\phi}_2^{(1,2)}(z), \\ \Omega_1^{(2,1)} \phi_1(z) &= L_\mathfrak{C}^{(2)}(z) \tilde{\phi}_1^{(2,1)}(z), & \Omega_2^{(2,1)} \tilde{\phi}_2^{(2,1)}(z) &= L_\Gamma^{(2)}(z) \phi_2(z). \end{aligned}$$

Definition 14. Given a Laurent polynomial $L(z)$ we consider $\delta L(z_1, z_2) := \frac{L(z_1) - L(z_2)}{z_1 - z_2}$ and the completely homogeneous symmetric polynomials $h_j(z_1, z_2) := (z_1)^j + (z_1)^{j-1}z_2 + \cdots + z_1(z_2)^{j-1} + (z_2)^j$ and their duals $h_j^*(z_1, z_2) := (z_1 z_2)^{-1} h_j((z_1)^{-1}, (z_2)^{-1})$ with $j \in \{0, 1, 2, \dots\}$.

Proposition 9. One has $\delta L(z_1, z_2) = \sum_{j=1}^n L_j h_{j-1}(z_1, z_2) - \sum_{j=1}^m L_{-j} h_{j-1}^*(z_1, z_2)$ and therefore the bivariate Laurent polynomial $\delta L(z_1, z_2)$ is symmetrical and, fixing one of the variables, is a Laurent polynomial in the other variable of positive maximum degree $n - 1$ and negative degree $-m$.

Proposition 10. The second kind functions are subject to the following connection formulas

$$(11) \quad (C_2(z))^\dagger (\Omega_2^{(1,2)})^\dagger - L_\Gamma^{(1)}(\bar{z}) (\tilde{C}_2^{(1,2)}(z))^\dagger = - \left\langle \delta L_\Gamma^{(1)}(z_1, \bar{z}), (\tilde{\Phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\bar{u}^{(1,2)}},$$

$$(12) \quad \Omega_1^{(2,1)} C_1(z) - \tilde{C}_1^{(2,1)}(z) \overline{L_\Gamma^{(2)}}(z) = - \left\langle \tilde{\Phi}_1^{(2,1)}(z_1), \delta L_\Gamma^{(2)}(z_2, \bar{z}) \right\rangle_{\bar{u}^{(2,1)}},$$

$$(13) \quad \Omega_1^{(1,2)} \tilde{C}_1^{(1,2)}(z) = \left\langle \Phi_1(z_1), \frac{L_\Gamma^{(2)}(z_2)}{\bar{z} - z_2} \right\rangle_u,$$

$$(14) \quad (\tilde{C}_2^{(2,1)}(z))^\dagger (\Omega_2^{(2)})^\dagger = \left\langle \frac{L_\Gamma^{(1)}(z_1)}{\bar{z} - z_1}, (\Phi_2(z_2))^\top \right\rangle_u.$$

Proposition 11. For $l \geq 2N_\Gamma$, the second kind functions satisfy the connection formulas

$$(15) \quad (\Omega_2^{(1,2)})_{l, l-2N_\Gamma} C_{2, l-2N_\Gamma}(z) + \cdots + (\Omega_2^{(1,2)})_{l, l+2N_\Gamma} C_{2, l+2N_\Gamma}(z) = \overline{L_\Gamma^{(1)}}(z) \tilde{C}_{2, l}^{(1,2)}(z),$$

$$(16) \quad (\Omega_1^{(2,1)})_{l, l-2N_\Gamma} C_{1, l-2N_\Gamma}(z) + \cdots + (\Omega_1^{(2,1)})_{l, l+2N_\Gamma} C_{1, l+2N_\Gamma}(z) = \overline{L_\Gamma^{(2)}}(z) \tilde{C}_{1, l}^{(2,1)}(z).$$

Proof. It is a consequence of the orthogonality relations in Corollary 1, because they involve

$$\left\langle \delta L_\Gamma^{(1)}(\bar{z}, z_1), (\tilde{\Phi}_{2, l}^{(1,2)}(z_2))^\top \right\rangle_{\bar{u}^{(1,2)}} = \left\langle \tilde{\Phi}_{1, l}^{(2,1)}(z_1), \delta L_\Gamma^{(2)}(\bar{z}, z_2) \right\rangle_{\bar{u}^{(2,1)}} = 0, \quad l \geq 2N_\Gamma.$$

□

2.2. Connection formulas for the Christoffel–Darboux kernels and their mixed versions.

Definition 15. Let's define the following upper triangular $2N_\Gamma \times 2N_\Gamma$ matrices

$$\Gamma_{2, l}^{(1,2)} := \begin{bmatrix} (\Omega_2^{(1,2)})_{l, l-2N_\Gamma} & (\Omega_2^{(1,2)})_{l, l-2N_\Gamma+1} & (\Omega_2^{(1,2)})_{l, l-2N_\Gamma+2} & \cdots & (\tilde{\Omega}_2^{(1,2)})_{l, l-1} \\ 0 & (\Omega_2^{(1,2)})_{l+1, l-2N_\Gamma+1} & (\Omega_2^{(1,2)})_{l+1, l-2N_\Gamma+2} & \cdots & (\Omega_2^{(1,2)})_{l+1, l-1} \\ 0 & 0 & (\Omega_2^{(1,2)})_{l+2, l-2N_\Gamma+1} & \cdots & (\Omega_2^{(1,2)})_{l+2, l-1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & (\Omega_2^{(1,2)})_{l+2N_\Gamma-1, l-1} \end{bmatrix},$$

$$\Gamma_{1, l}^{(2,1)} := \begin{bmatrix} (\Omega_1^{(2,1)})_{l, l-2N_\Gamma} & (\Omega_1^{(2,1)})_{l, l-2N_\Gamma+1} & (\Omega_1^{(2,1)})_{l, l-2N_\Gamma+2} & \cdots & (\Omega_1^{(2,1)})_{l, l-1} \\ 0 & (\Omega_1^{(2,1)})_{l+1, l-2N_\Gamma+1} & (\Omega_1^{(2,1)})_{l+1, l-2N_\Gamma+2} & \cdots & (\Omega_1^{(2,1)})_{l+1, l-1} \\ 0 & 0 & (\Omega_1^{(2,1)})_{l+2, l-2N_\Gamma+1} & \cdots & (\Omega_1^{(2,1)})_{l+2, l-1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & (\Omega_1^{(2,1)})_{l+2N_\Gamma-1, l-1} \end{bmatrix},$$

and the following lower triangular $2\underline{N}_e \times 2\underline{N}_e$ matrices

$$\mathfrak{C}_{2,l}^{(1,2)} := \begin{bmatrix} (\Omega_2^{(1,2)})_{l-2\underline{N}_e,l} & 0 & 0 & \dots & 0 \\ (\Omega_2^{(1,2)})_{l-2\underline{N}_e+1,l} & (\Omega_2^{(1,2)})_{l-2\underline{N}_e+1,l+1} & 0 & \dots & 0 \\ (\Omega_2^{(1,2)})_{l-2\underline{N}_e+2,l} & (\Omega_2^{(1,2)})_{l-2\underline{N}_e+2,l+1} & (\Omega_2^{(1,2)})_{l-2\underline{N}_e+2,l+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\Omega_2^{(1,2)})_{l-1,l} & (\Omega_2^{(1,2)})_{l-1,l+1} & (\Omega_2^{(1,2)})_{l-1,l+2} & \dots & (\Omega_2^{(1,2)})_{l-1,l+2\underline{N}_e-1} \end{bmatrix},$$

$$\mathfrak{C}_{1,l}^{(2,1)} := \begin{bmatrix} (\Omega_1^{(2,1)})_{l-2\underline{N}_e,l} & 0 & 0 & \dots & 0 \\ (\Omega_1^{(2,1)})_{l-2\underline{N}_e+1,l} & (\Omega_1^{(2,1)})_{l-2\underline{N}_e+1,l+1} & 0 & \dots & 0 \\ (\Omega_1^{(2,1)})_{l-2\underline{N}_e+2,l} & (\Omega_1^{(2,1)})_{l-2\underline{N}_e+2,l+1} & (\Omega_1^{(2,1)})_{l-2\underline{N}_e+2,l+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\Omega_1^{(2,1)})_{l-1,l} & (\Omega_1^{(2,1)})_{l-1,l+1} & (\Omega_1^{(2,1)})_{l-1,l+2} & \dots & (\Omega_1^{(2,1)})_{l-1,l+2\underline{N}_e-1} \end{bmatrix}.$$

Proposition 12. For $l \geq 2 \max(\underline{N}_e, \underline{N}_\Gamma)$, the Christoffel–Darboux kernels and their Geronimus–Uvarov transformations satisfy the following connection formulas

$$(17) \quad \overline{L_e^{(2)}(z_1)} \tilde{K}^{(1,2),[l]}(\bar{z}_1, z_2) - L_\Gamma^{(1)}(z_2) K^{[l]}(\bar{z}_1, z_2) \\ = \left[\tilde{\Phi}_{1,l-2\underline{N}_e}^{(1,2)}(z_2) (\tilde{H}_{l-2\underline{N}_e}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2\underline{N}_\Gamma-1}^{(1,2)}(z_2) (\tilde{H}_{l+2\underline{N}_\Gamma-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2\underline{N}_e \times 2\underline{N}_\Gamma} & \overline{\mathfrak{C}_{2,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2\underline{N}_\Gamma \times 2\underline{N}_e} \end{bmatrix} \begin{bmatrix} \overline{\Phi_{2,l-2\underline{N}_\Gamma}(z_1)} \\ \vdots \\ \overline{\Phi_{2,l+2\underline{N}_e-1}(z_1)} \end{bmatrix},$$

$$(18) \quad L_e^{(1)}(z_2) \tilde{K}^{(2,1),[l]}(\bar{z}_1, z_2) - \overline{L_\Gamma^{(2)}(z_1)} K^{[l]}(\bar{z}_1, z_2) \\ = \left[\overline{\tilde{\Phi}_{2,l-2\underline{N}_e}^{(2,1)}(z_1)} (\tilde{H}_{l-2\underline{N}_e}^{(2,1)})^{-1}, \dots, \overline{\tilde{\Phi}_{2,l+2\underline{N}_\Gamma-1}^{(2,1)}(z_1)} (\tilde{H}_{l+2\underline{N}_\Gamma-1}^{(2,1)})^{-1} \right] \begin{bmatrix} 0_{2\underline{N}_e \times 2\underline{N}_\Gamma} & \mathfrak{C}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2\underline{N}_\Gamma \times 2\underline{N}_e} \end{bmatrix} \begin{bmatrix} \Phi_{1,l-2\underline{N}_\Gamma}(z_2) \\ \vdots \\ \Phi_{1,l+2\underline{N}_e-1}(z_2) \end{bmatrix}.$$

Proposition 13. For $l \geq 2 \max(\underline{N}_e, \underline{N}_\Gamma)$, under Geronimus–Uvarov transformations the mixed Christoffel–Darboux kernels satisfy the connection formulas

$$(19) \quad L_\Gamma^{(1)}(\bar{x}_1) \tilde{K}_{C_2}^{(1,2),[l]}(\bar{x}_1, x_2) - L_\Gamma^{(1)}(x_2) K_{C_2}^{[l]}(\bar{x}_1, x_2) - \delta L_\Gamma^{(1)}(\bar{x}_1, x_2) \\ = \left[\tilde{\Phi}_{1,l-2\underline{N}_e}^{(1,2)}(x_2) (\tilde{H}_{l-2\underline{N}_e}^{(1)})^{-1}, \dots, \tilde{\Phi}_{1,l+2\underline{N}_\Gamma-1}^{(1,2)}(x_2) (\tilde{H}_{l+2\underline{N}_\Gamma-1}^{(1)})^{-1} \right] \begin{bmatrix} 0_{2\underline{N}_e \times 2\underline{N}_\Gamma} & \overline{\mathfrak{C}_{2,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2\underline{N}_\Gamma \times 2\underline{N}_e} \end{bmatrix} \begin{bmatrix} \overline{C_{2,l-2\underline{N}_\Gamma}(x_1)} \\ \vdots \\ \overline{C_{2,l+2\underline{N}_e-1}(x_1)} \end{bmatrix},$$

$$(20) \quad L_e^{(1)}(x_2) \tilde{K}_{C_2}^{(2,1),[l]}(\bar{x}_1, x_2) - \left\langle \frac{L_e^{(1)}(z_1)}{\bar{x}_1 - z_1}, \overline{K^{[l]}(\bar{z}_2, x_2)} \right\rangle_u \\ = \left[\tilde{C}_{2,l-2\underline{N}_e}^{(2,1)}(x_1) (\tilde{H}_{l-2\underline{N}_e}^{(2,1)})^{-1}, \dots, \tilde{C}_{2,l+2\underline{N}_\Gamma-1}^{(2,1)}(x_1) (\tilde{H}_{l+2\underline{N}_\Gamma-1}^{(2,1)})^{-1} \right] \begin{bmatrix} 0_{2\underline{N}_e \times 2\underline{N}_\Gamma} & \mathfrak{C}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2\underline{N}_\Gamma \times 2\underline{N}_e} \end{bmatrix} \begin{bmatrix} \Phi_{1,l-2\underline{N}_\Gamma}(x_2) \\ \vdots \\ \Phi_{1,l+2\underline{N}_e-1}(x_2) \end{bmatrix}$$

(21)

$$\begin{aligned} & \overline{L_{\mathbf{e}}^{(2)}(x_1)} \tilde{K}_{C_1}^{(1,2),[1]}(\bar{x}_1, x_2) - \left\langle K^{[1]}(\bar{x}_1, z_1), \frac{L_{\mathbf{e}}^{(2)}(z_2)}{\bar{x}_2 - z_2} \right\rangle_u \\ &= \left[\tilde{C}_{1,l-2N_{\mathbf{e}}}^{(1,2)}(x_2)(\tilde{H}_{l-2N_{\mathbf{e}}}^{(1)})^{-1}, \dots, \tilde{C}_{1,l+2N_{\Gamma}-1}^{(1,2)}(x_2)(\tilde{H}_{l+2N_{\Gamma}-1}^{(1)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathbf{e}} \times 2N_{\Gamma}} & \overline{\mathbf{e}_{2,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathbf{e}}} \end{bmatrix} \begin{bmatrix} \overline{\phi_{2,l-2N_{\Gamma}}(x_1)} \\ \vdots \\ \overline{\phi_{2,l+2N_{\mathbf{e}}-1}(x_1)} \end{bmatrix}, \end{aligned}$$

(22)

$$\begin{aligned} & \overline{L_{\Gamma}^{(2)}(x_2)} \tilde{K}_{C_1}^{(2,1),[1]}(\bar{x}_1, x_2) - \overline{L_{\Gamma}^{(2)}(\bar{x}_1)} K_{C_1}^{[1]}(\bar{x}_1, x_2) - \overline{\delta L_{\Gamma}^{(2)}(\bar{x}_1, x_2)} \\ &= \left[\overline{\tilde{\phi}_{2,l-2N_{\mathbf{e}}}^{(2,1)}(x_1)(\tilde{H}_{l-2N_{\mathbf{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1)(\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathbf{e}} \times 2N_{\Gamma}} & \mathbf{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathbf{e}}} \end{bmatrix} \begin{bmatrix} C_{1,l-2N_{\Gamma}}(x_2) \\ \vdots \\ C_{1,l+2N_{\mathbf{e}}-1}(x_2) \end{bmatrix}. \end{aligned}$$

2.3. Christoffel formulas for Geronimus–Uvarov transformations. The Geronimus–Uvarov perturbed second kind functions are

$$\overline{\tilde{C}_{2,k}^{(1,2)}(z)} = \left\langle \frac{1}{\bar{z} - z_1}, \tilde{\phi}_{2,k}^{(1,2)}(z_2) \right\rangle_{\frac{L_{\mathbf{e}}^{(2)}(z_2)}{L_{\Gamma}^{(1)}(z_1)} u} + \sum_{i=1}^{d(1)} \sum_{l=0}^{m_i^{(1)}-1} \frac{1}{l!} \frac{d^l}{d\zeta^l} \left(\frac{1}{\bar{z} - \zeta} \right) \Big|_{\zeta=\zeta_i^{(1)}} \overline{\langle L_{\mathbf{e}}^{(2)} \xi_{i,l}^{(1)}, \tilde{\phi}_{2,k}^{(1,2)} \rangle},$$

for $\bar{z} \in \mathbb{C} \setminus \text{supp}_1(u) \cup \sigma(L_{\Gamma}^{(1)})$ and

$$\tilde{C}_{1,k}^{(2,1)}(z) = \left\langle \tilde{\phi}_{1,k}^{(2,1)}(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_{\frac{L_{\mathbf{e}}^{(1)}(z_1)}{L_{\Gamma}^{(2)}(z_2)} u} + \sum_{i=1}^{d(2)} \sum_{l=0}^{m_i^{(2)}-1} \left\langle L_{\mathbf{e}}^{(1)} \xi_{i,l}^{(2)}, \tilde{\phi}_{1,k}^{(2,1)} \right\rangle \frac{1}{l!} \frac{d^l}{d\zeta^l} \left(\frac{1}{\bar{z} - \zeta} \right) \Big|_{\zeta=\zeta_i^{(2)}},$$

for $z \in \mathbb{C} \setminus \overline{\text{supp}_2(u)} \cup \sigma(L_{\Gamma}^{(2)})$.

Definition 16. i) For $\alpha \in \{1, 2\}$ we define

$$\langle \xi_i^{(\alpha)}, f \rangle := \left[\langle \xi_{i,1}^{(\alpha)}, f \rangle, \dots, \langle \xi_{i,m_i-1}^{(\alpha)}, f \rangle \right], \quad \langle \xi^{(\alpha)}, f \rangle := \left[\langle \xi_1^{(\alpha)}, f \rangle, \dots, \langle \xi_d^{(\alpha)}, f \rangle \right].$$

ii) The expression $L(z) = L_{N+z} z^{-N^-} \prod_{j=1}^d (z - \zeta_j)^{m_j}$ with $m_1 + \dots + m_d = N^+ + N^-$, allows us to introduce

$$L_{[i]}(z) := L_{N+z} z^{-N^-} \prod_{\substack{j=1 \\ j \neq i}}^d (z - \zeta_j)^{m_j}.$$

Relations (16) and (15) motivate us to consider

$$\begin{aligned} \overline{L_{\Gamma}^{(1)}(z)} \tilde{C}_{2,k}^{(1,2)}(z) &= \left\langle \frac{L_{\Gamma}^{(1)}(\bar{z})}{\bar{z} - z_1}, \tilde{\phi}_{2,k}^{(1,2)}(z_2) \right\rangle_{\frac{L_{\mathbf{e}}^{(2)}(z_2)}{L_{\Gamma}^{(1)}(z_1)} u} + \sum_{i=1}^{d(1)} \overline{\langle L_{\mathbf{e}}^{(2)} \xi_i^{(1)}, \tilde{\phi}_{2,k}^{(1,2)} \rangle} L_{\Gamma,[i]}^{(1)}(\bar{z}) \begin{bmatrix} (\bar{z} - \zeta_i^{(1)})^{m_i^{(1)}-1} \\ (\bar{z} - \zeta_i^{(1)})^{m_i^{(1)}-2} \\ \vdots \\ 1 \end{bmatrix}, \\ \overline{L_{\Gamma}^{(2)}(z)} \tilde{C}_{1,k}^{(2,1)}(z) &= \left\langle \tilde{\phi}_{1,k}^{(2,1)}(z_1), \frac{L_{\Gamma}^{(2)}(\bar{z})}{\bar{z} - z_2} \right\rangle_{\frac{L_{\mathbf{e}}^{(1)}(z_1)}{L_{\Gamma}^{(2)}(z_2)} u} + \sum_{i=1}^{d(2)} \overline{\langle L_{\mathbf{e}}^{(1)} \xi_i^{(2)}, \tilde{\phi}_{1,k}^{(2,1)} \rangle} \overline{L_{\Gamma,[i]}^{(2)}(z)} \begin{bmatrix} (z - \bar{\zeta}_i^{(2)})^{m_i^{(2)}-1} \\ (z - \bar{\zeta}_i^{(2)})^{m_i^{(2)}-2} \\ \vdots \\ 1 \end{bmatrix}. \end{aligned}$$

When the evaluation of the spectral jets $\overline{\mathcal{J}_{\Gamma}^{(1)}}_{\overline{L_{\Gamma}^{(1)}} \tilde{C}_{2,k}^{(1,2)}}$ y $\overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{L_{\Gamma}^{(2)}} \tilde{C}_{1,k}^{(2,1)}}$ is required we perform it by taking appropriated limits —and in this manner we take care of the fact that the perturbing polynomial zeros lay on the border of the perturbed functional support.

Definition 17. For $\alpha \in \{1, 2\}$, $j \in \{1, \dots, d^{(\alpha)}\}$, $k \in \{0, \dots, m_j^{(\alpha)} - 1\}$, we define

$$\ell_{[j],k}^{(\alpha)} := \frac{1}{k!} \frac{d^k \overline{L_{\Gamma,[j]}^{(\alpha)}}(z)}{dz^k} \Big|_{z=\tilde{z}_j^{(\alpha)}}.$$

We also introduce

$$\mathcal{L}_{[j]}^{(\alpha)} := \begin{bmatrix} 0 & 0 & 0 & \dots & \ell_{[j],0}^{(\alpha)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ell_{[j],0}^{(\alpha)} & \dots & \ell_{[j],m_j^{(\alpha)}-3}^{(\alpha)} \\ 0 & \ell_{[j],0}^{(\alpha)} & \ell_{[j],1}^{(\alpha)} & \dots & \ell_{[j],m_j^{(\alpha)}-2}^{(\alpha)} \\ \ell_{[j],0}^{(\alpha)} & \ell_{[j],1}^{(\alpha)} & \ell_{[j],2}^{(\alpha)} & \dots & \ell_{[j],m_j^{(\alpha)}-1}^{(\alpha)} \end{bmatrix}, \quad \mathcal{L}^{(\alpha)} := \text{diag}(\mathcal{L}_{[1]}^{(\alpha)}, \dots, \mathcal{L}_{[d^{(\alpha)}]}^{(\alpha)}).$$

Proposition 14. The spectral jets fulfill

$$(23) \quad \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{L_{\Gamma}^{(2)}} \tilde{C}_{1,k}^{(2,1)}} = \langle L_{\mathfrak{e}}^{(1)} \xi^{(2)}, \tilde{\Phi}_{1,k}^{(2,1)} \rangle \mathcal{L}^{(2)},$$

$$(24) \quad \overline{\mathcal{J}_{\Gamma}^{(1)}}_{\overline{L_{\Gamma}^{(1)}} \tilde{C}_{2,k}^{(1,2)}} = \langle L_{\mathfrak{e}}^{(2)} \xi^{(1)}, \tilde{\Phi}_{2,k}^{(1,2)} \rangle \mathcal{L}^{(1)}.$$

Theorem 1 (Christoffel–Geronimus–Uvarov formulas). Let's assume that all perturbing Laurent polynomials are prepared and consider the determinants

$$\tilde{\tau}_l^{(2,1)} := \begin{vmatrix} \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l-2N_{\Gamma}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \mathcal{J}_{\Phi_{1,l-2N_{\Gamma}}}^{(1)} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l+2N_{\mathfrak{e}}-1}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{(1)} \end{vmatrix}, \quad \tilde{\tau}_l^{(1,2)} := \begin{vmatrix} \overline{\mathcal{J}_{\Gamma}^{(1)}}_{\overline{C_{2,l-2N_{\Gamma}}} - \langle \xi^{(1)}, \Phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} & \mathcal{J}_{\Phi_{2,l-2N_{\Gamma}}}^{(2)} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{\Gamma}^{(1)}}_{\overline{C_{2,l+2N_{\mathfrak{e}}-1}}} - \langle \xi^{(1)}, \Phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\Phi_{2,l+2N_{\mathfrak{e}}-1}}^{(2)} \end{vmatrix}.$$

Then, for $l \geq 2 \max(N_{\Gamma}, N_{\mathfrak{e}})$ and $\tilde{\tau}_l^{(2,1)} \neq 0$ we have the following determinantal formulas

$$(25) \quad \tilde{\Phi}_{1,l}^{(2,1)}(z) = \frac{L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)}}{L_{\mathfrak{e}}^{(1)}(z)} \frac{1}{\tilde{\tau}_l^{(2,1)}} \begin{vmatrix} \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l-2N_{\Gamma}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \mathcal{J}_{\Phi_{1,l-2N_{\Gamma}}}^{(1)} & \Phi_{1,l-2N_{\Gamma}}(z) \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l+2N_{\mathfrak{e}}}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{(1)} & \Phi_{1,l+2N_{\mathfrak{e}}}(z) \end{vmatrix},$$

$$(26) \quad \tilde{H}_l^{(2,1)} = \frac{L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)}}{L_{\Gamma,(-1)^{lN_{\Gamma}}}^{(2)}} H_{l-2N_{\Gamma}} \frac{\tilde{\tau}_{l+1}^{(2,1)}}{\tilde{\tau}_l^{(2,1)}},$$

(27)

$$\tilde{\Phi}_{2,l}^{(2,1)}(z) = -\frac{\overline{L_{\Gamma}^{(2)}}(z)}{L_{\Gamma,(-1)^{lN_{\Gamma}}}^{(2)}} \frac{H_{l-2N_{\Gamma}}}{\tilde{\tau}_l^{(2,1)}} \begin{vmatrix} \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l-2N_{\Gamma}+1}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_{\Gamma}+1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_{\Gamma}+1}}^{(1)} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{C_{1,l+2N_{\mathfrak{e}}-1}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{(1)} \\ \overline{\mathcal{J}_{\Gamma}^{(2)}}_{\overline{K_{C_1}^{[l-2N_{\Gamma}+1]}}}(\bar{z}) - \langle (\xi^{(2)})_w, K^{[l-2N_{\Gamma}+1]}(\bar{z}, w) \rangle \mathcal{L}^{(2)} + \frac{1}{L_{\Gamma}^{(2)}(z)} \overline{\mathcal{J}_{\Gamma}^{(2)}}(\bar{z}) & \mathcal{J}_{K^{[l-2N_{\Gamma}+1]}}^{(1)}(\bar{z}) \end{vmatrix}$$

where the spectral of the mixed Christoffel–Darboux kernel and of $\delta L_\Gamma^{(2)}$ are taken with respect to the second variable. Whenever $l \geq 2 \max(N_\Gamma, N_\epsilon)$ and $\tilde{\tau}_l^{(1,2)} \neq 0$ the following formulas are satisfied

$$(28) \quad \tilde{\Phi}_{2,l}^{(1,2)}(z) = \frac{L_{\epsilon,(-1)^l N_\epsilon}^{(2)}}{L_\epsilon^{(2)}(z)} \frac{1}{\tilde{\tau}_l^{(1,2)}} \begin{vmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l-2N_\Gamma} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\Phi_{2,l-2N_\Gamma}}^{L_\epsilon^{(2)}} & \Phi_{2,l-2N_\Gamma}(z) \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_\epsilon}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l+2N_\epsilon} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\Phi_{2,l+2N_\epsilon}}^{L_\epsilon^{(2)}} & \Phi_{2,l+2N_\epsilon}(z) \end{vmatrix}$$

$$(29) \quad \tilde{H}_l^{(1,2)} = \frac{L_{\epsilon,(-1)^l N_\epsilon}^{(2)}}{L_{\Gamma,(-1)^l N_\Gamma}^{(2)}} \tilde{H}_{l-2N_\Gamma} \frac{\tilde{\tau}_{l+1}^{(1,2)}}{\tilde{\tau}_l^{(1,2)}},$$

$$(30) \quad \tilde{\Phi}_{1,l}^{(1,2)}(z) = - \frac{L_\Gamma^{(1)}(z)}{L_{\Gamma,(-1)^l N_\Gamma}^{(1)}} \frac{H_{l-2N_\Gamma}}{\tilde{\tau}_l^{(1,2)}} \begin{vmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma+1}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l-2N_\Gamma+1} \rangle \mathcal{L}^{(1)} & \overline{\mathcal{J}_{\Phi_{2,l-2N_\Gamma+1}}^{L_\epsilon^{(2)}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_\epsilon-1}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l+2N_\epsilon-1} \rangle \mathcal{L}^{(1)} & \overline{\mathcal{J}_{\Phi_{2,l+2N_\epsilon-1}}^{L_\epsilon^{(2)}}} \\ \overline{\mathcal{J}_{K_{C_2}^{[l-2N_\Gamma+1]}}^{L_\Gamma^{(1)}}}(z) - \langle (\xi^{(1)})_w, K^{[l-2N_\Gamma+1]}(\bar{w}, z) \rangle \overline{\mathcal{L}^{(1)}} + \frac{1}{L_\Gamma^{(1)}(z)} \overline{\mathcal{J}_{\delta L_\Gamma^{(1)}}^{L_\Gamma^{(1)}}}(z) & \overline{\mathcal{J}_{K^{[l-2N_\Gamma+1]}}^{L_\epsilon^{(2)}}}(z) \end{vmatrix},$$

where the spectral jet of the Christoffel–Darboux kernels and of $\delta L_\Gamma^{(1)}$ are taken with respect to the first variable.

3. REDUCTIONS

A bivariate linear functional u_{z_1, \bar{z}_2} is supported on the diagonal $z_1 = z_2$ if

$$\langle f(z_1), g(z_2) \rangle_u = \sum_{0 \leq n, m \ll \infty} \left\langle u_z^{(n,m)}, \frac{\partial^n f}{\partial z^n}(z) \frac{\partial^m g}{\partial z^m}(z) \right\rangle,$$

where $u_z^{(n,m)}$ are univariate linear functionals, i.e., we are dealing with a Sobolev sesquilinear form. A particularly relevant example, is the zero order diagonal case

$$\langle f(z), g(z) \rangle_u = \langle u_z, f(z) \overline{g(z)} \rangle.$$

The Toeplitz case appears, for zero order diagonal situations, when the support of the liner functional lays in the unit circle, $\text{supp } u \subset \mathbb{T}$.¹

For these cases, we have $g_{n,m} = \langle u_z, z^{n-m} \rangle$, which happens to be a Toeplitz matrix, $g_{n,m} = g_{n+1,m+1}$. In this Toeplitz scenario the CMV Gram matrix has the following moment matrix form $G = \langle u_z, \chi(z) (\chi(z))^\dagger \rangle$.

Another important reduction appears for the zero order diagonal case when the linear functional u_z happens to be, not only quasidefinite, but also real, i.e. $\overline{\langle u_z, f(z) \rangle} = \langle u_z, \overline{f(z)} \rangle$ for every test function $f(z) \in \mathcal{F}$. Then, the Gram matrix is also Hermitian $G = G^\dagger$ and, consequently, $S_1 = S_2$ and $H = \bar{H}$. For non-negative linear functionals u_z ($\langle u_z, f(z) \rangle \in \mathbb{R}^+$, for every real test function $f : \mathbb{C} \rightarrow \mathbb{R}^+$). For real linear functionals the biorthogonality collapses to pseudo-orthogonality (or quasidefinite orthogonality), and for non-negative linear functionals to orthogonality, in this case we have $H_l > 0$, $l \in \{0, 1, \dots\}$.

Féjer [22] and Riesz [37] found a representation for nonnegative trigonometric polynomials. Nonnegative trigonometric polynomials of the form $f(\theta) = a_0 + \sum_{k=1}^n (a_k \cos(k\theta) + b_k \sin(k\theta))$ can always be

¹For higher orders diagonal situations supported in the unit circle we can also find Toeplitz matrices. Indeed, that is the case for the following linear functional $\langle f(z), g(z) \rangle_u = \langle u_z, z \frac{\partial f}{\partial z}(z) \overline{g(z)} \rangle - \langle u_z, f(z) z \frac{\partial g}{\partial z}(z) \rangle$ where u_z is a functional supported in the unit circle.

written as $f(\theta) = |p(z)|^2$ where $p(z) = \sum_{l=0}^n p_n z^n$ and $z = e^{i\theta}$. This is equivalent to write [29] $f(\theta) = z^{-n}P(z)$ with $z = e^{i\theta}$ and $P(z) \in \mathbb{C}[z]$ is a polynomial with $\deg P(z) = 2n$ such that $P(z) = P^*(z)$, fulfilling $z^{-n}P(z) = |P(z)|$ for $z \in \mathbb{T}$. The Szegő reciprocal polynomial is $P^*(z) := z^{2n}\bar{P}(z^{-1})$ of $P(z)$. Observe that for $z \in \mathbb{C}^*$ the function $L(z) = z^{-n}P(z)$ is not any more a trigonometric polynomial but a Laurent polynomial. Given a Laurent polynomial its reciprocal is given by $L_*(z) := \bar{L}(z^{-1})$, thus for $L(z) = z^{-n}P(z)$ we have $L_*(z) = z^n\bar{P}(z^{-1}) = z^{-n}P^*(z)$ and if $P(z) = P^*(z)$ we find $L_*(z) = L(z)$; the positivity condition reads: $f(\theta) = L(z)$ with $L(z)$ a Laurent polynomial with $L(z) = L_*(z)$ and $L(z) = |L(z)|$ for $z \in \mathbb{T}$. Notice that the self-reciprocal condition $L(z) = L_*(z)$ is simply the reality condition for the corresponding Laurent polynomial on the unit circle, that is $L(z) = \overline{L(z)}$ for $z \in \mathbb{T}$, or equivalently that the corresponding trigonometric polynomial takes real values.

The Geronimus–Uvarov perturbations within a zero order Toeplitz scenario are of the form

$$\begin{aligned}\tilde{u}_{z_1, \bar{z}_2} &= \frac{L_{\mathfrak{C},*}^{(2)}(z)}{L_{\Gamma}^{(1)}(z)}(u)_z + \overline{L_{\mathfrak{C}}^{(2)}(z_2)} \sum_{i=1}^{d^{(1)}} \sum_{l=0}^{m_i^{(1)}-1} \frac{(-1)^l}{l!} \delta^{(l)}(z_1 - \zeta_i^{(1)}) \otimes \overline{(\xi_{i,l}^{(1)})_{z_2}}, \\ \tilde{u}_{z_1, \bar{z}_2} &= \frac{L_{\mathfrak{C}}^{(1)}(z)}{L_{\Gamma,*}^{(2)}(z)}(u)_z + L_{\mathfrak{C}}^{(1)}(z_1) \sum_{i=1}^{d^{(2)}} \sum_{l=0}^{m_i^{(2)}-1} \frac{(-1)^l}{l!} (\xi_{i,l}^{(2)})_{z_1} \otimes \overline{\delta^{(l)}(z_2 - \zeta_i^{(2)})}.\end{aligned}$$

When do these two transformations happen to be same? Let us assume that $L_{\mathfrak{C},*}^{(2)}(z)$ and $L_{\Gamma}^{(1)}(z)$ are coprime Laurent polynomials and that $L_{\mathfrak{C}}^{(1)}(z)$ and $L_{\Gamma,*}^{(2)}(z)$ are coprime, as well. Thus, both transformations could possibly be the same if only if

$$L_{\mathfrak{C},*}^{(2)}(z) = L_{\mathfrak{C}}^{(1)}(z) =: L_{\mathfrak{C}}(z), \quad L_{\Gamma}^{(1)}(z) = L_{\Gamma,*}^{(2)}(z) =: L_{\Gamma}(z).$$

We will take the following linear functionals $L_{\mathfrak{C},*}(z_2)(\xi_{i,k}^{(1)})_{z_2} = \sum_{j=1}^d \sum_{l=0}^{m_j-1} \frac{(-1)^l}{l!} \bar{\Xi}_{i,k|j,l} \delta^{(l)}(z_2 - (\bar{\zeta}_j)^{-1})$ and $L_{\mathfrak{C}}(z)(\xi_{j,l}^{(2)})_{z_1} = \sum_{i=1}^d \sum_{k=0}^{m_i-1} \frac{(-1)^k}{k!} \Xi_{i,k|j,l} \delta^{(k)}(z_1 - \zeta_i)$, with ζ_i, m_i the zeros and corresponding multiplicities of L_{Γ} , and the Geronimus–Uvarov transforms are

$$\tilde{u}_{z_1, \bar{z}_2} = \frac{L_{\mathfrak{C}}(z)}{L_{\Gamma}(z)}(u)_z + \sum_{i,j=1}^d \sum_{k=0}^{m_i-1} \sum_{l=0}^{m_j-1} \frac{(-1)^{k+l}}{k!l!} \bar{\Xi}_{i,k|j,l} \delta^{(k)}(z_1 - \zeta_i) \otimes \overline{\delta^{(l)}(z_2 - (\bar{\zeta}_j)^{-1})},$$

so that

$$\langle f(z_1), g(z_2) \rangle_{\tilde{u}} = \left\langle \frac{L_{\mathfrak{C}}(z)}{L_{\Gamma}(z)}(u)_z, f(z) \overline{g(z)} \right\rangle + \sum_{i,j=1}^d \sum_{k=0}^{m_i-1} \sum_{l=0}^{m_j-1} \frac{1}{k!l!} \bar{\Xi}_{i,k|j,l} f^{(k)}(\zeta_i) \overline{g^{(l)}((\bar{\zeta}_j)^{-1})}.$$

This mass term is possibly not supported neither on the diagonal nor on the unit circle and, therefore, not Toeplitz. However, it is the most general mass term such that both Geronimus–Uvarov transformations, of a zero order Toeplitz sesquilinear form, are equal. We observe that [12]

$$\begin{aligned}\langle (\xi_{i,k}^{(1)})_{z_2}, \phi(z_2) \rangle &= \sum_{j=1}^d \sum_{l=0}^{m_j-1} \frac{1}{l!} \left(\frac{\phi(\zeta)}{L_{\mathfrak{C},*}(\zeta)} \right)^{(l)} \Big|_{\zeta=(\bar{\zeta}_j)^{-1}} \bar{\Xi}_{i,k|j,l}, \\ \langle (\xi_{j,l}^{(2)})_{z_1}, \phi(z_1) \rangle &= \sum_{i=1}^d \sum_{k=0}^{m_i-1} \frac{1}{k!} \left(\frac{\phi(\zeta)}{L_{\mathfrak{C}}(\zeta)} \right)^{(k)} \Big|_{\zeta=\zeta_i} \Xi_{i,k|j,l},\end{aligned}$$

and introduce

$$\Xi := \begin{bmatrix} \Xi_{1,0|1,0} & \cdots & \Xi_{1,0|1,m_1-1} & \cdots & \Xi_{1,0|d,0} & \cdots & \Xi_{1,0|d,m_d-1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \Xi_{1,m_1-1|1,0} & \cdots & \Xi_{1,m_1-1|1,m_1-1} & \cdots & \Xi_{1,m_1-1|d,0} & \cdots & \Xi_{1,m_1-1|d,m_d-1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \Xi_{d,0|1,0} & \cdots & \Xi_{d,0|1,m_1-1} & \cdots & \Xi_{d,0|d,0} & \cdots & \Xi_{d,0|d,m_d-1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \Xi_{d,m_d-1|1,0} & \cdots & \Xi_{d,m_d-1|1,m_1-1} & \cdots & \Xi_{d,m_d-1|d,0} & \cdots & \Xi_{d,m_d-1|d,m_d-1} \end{bmatrix} \in \mathbb{C}^{2n \times 2n}.$$

Then, the following expressions in terms of spectral jets $\langle \xi^{(1)}, \phi \rangle = \mathcal{J}_{\frac{\phi}{L_{\mathcal{C}}^*}}^{\overline{L_{\Gamma}^*}} \Xi^\dagger$ and $\langle \xi^{(2)}, \phi \rangle = \mathcal{J}_{\frac{\phi}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}}$ hold. In this setting it is convenient to introduce

$$\tilde{\tau}_l^{(2,1)} := \begin{vmatrix} \mathcal{J}_{C_{1,l-2N_\Gamma}}^{\overline{L_{\Gamma}^*}} - \mathcal{J}_{\frac{\phi_{1,l-2N_\Gamma}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\phi_{1,l-2N_\Gamma}}^{L_{\mathcal{C}}} \\ \vdots & \vdots \\ \mathcal{J}_{C_{1,l+2N_{\mathcal{C}}-1}}^{\overline{L_{\Gamma}^*}} - \mathcal{J}_{\frac{\phi_{1,l+2N_{\mathcal{C}}-1}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\phi_{1,l+2N_{\mathcal{C}}-1}}^{L_{\mathcal{C}}} \end{vmatrix}, \quad \tilde{\tau}_l^{(1,2)} := \begin{vmatrix} \mathcal{J}_{C_{2,l-2N_\Gamma}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\phi_{2,l-2N_\Gamma}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma}^*}} \Xi^\dagger \mathcal{L} & \mathcal{J}_{\phi_{2,l-2N_\Gamma}}^{L_{\mathcal{C},*}} \\ \vdots & \vdots \\ \mathcal{J}_{C_{2,l+2N_{\mathcal{C}}-1}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\phi_{2,l+2N_{\mathcal{C}}-1}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma}^*}} \Xi^\dagger \mathcal{L} & \mathcal{J}_{\phi_{2,l+2N_{\mathcal{C}}-1}}^{L_{\mathcal{C},*}} \end{vmatrix}.$$

Proposition 15 (Geronimus–Uvarov perturbations of the zero order Toeplitz case). *Let's assume that all perturbing Laurent polynomials are prepared. Then, for $l \geq 2 \max(N_\Gamma, N_{\mathcal{C}})$ and $\tilde{\tau}_l^{(2,1)} \tilde{\tau}_l^{(1,2)} \neq 0$ we have the following determinantal formulas*

$$\begin{aligned} \tilde{\Phi}_{1,l}(z) &= \frac{L_{\mathcal{C},(-1)^{lN_{\mathcal{C}}}}}{L_{\mathcal{C}}(z)} \frac{1}{\tilde{\tau}_l^{(2,1)}} \begin{vmatrix} \mathcal{J}_{C_{1,l-2N_\Gamma}}^{\overline{L_{\Gamma}^*}} - \mathcal{J}_{\frac{\phi_{1,l-2N_\Gamma}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\phi_{1,l-2N_\Gamma}}^{L_{\mathcal{C}}} & \phi_{1,l-2N_\Gamma}(z) \\ \vdots & \vdots & \vdots \\ \mathcal{J}_{C_{1,l+2N_{\mathcal{C}}}}^{\overline{L_{\Gamma}^*}} - \mathcal{J}_{\frac{\phi_{1,l+2N_{\mathcal{C}}}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\phi_{1,l+2N_{\mathcal{C}}}}^{L_{\mathcal{C}}} & \phi_{1,l+2N_{\mathcal{C}}}(z) \end{vmatrix} \\ &= -\frac{L_\Gamma(z)}{L_{\Gamma,(-1)^{lN_\Gamma}}} \frac{H_{l-2N_\Gamma}}{\tilde{\tau}_l^{(1,2)}} \begin{vmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma+1}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\phi_{2,l-2N_\Gamma+1}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma}^*}} \Xi^\dagger \mathcal{L}} & \overline{\mathcal{J}_{\phi_{2,l-2N_\Gamma+1}}^{L_{\mathcal{C},*}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathcal{C}}-1}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\phi_{2,l+2N_{\mathcal{C}}-1}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma}^*}} \Xi^\dagger \mathcal{L}} & \overline{\mathcal{J}_{\phi_{2,l+2N_{\mathcal{C}}-1}}^{L_{\mathcal{C},*}}} \\ \mathcal{J}_{K_{C_2}^{[l-2N_\Gamma+1]}}^{\overline{L_{\Gamma}}} (z) - \mathcal{J}_{\frac{K^{[l-2N_\Gamma+1]}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma}^*}} \Xi^\top \overline{\mathcal{L}} + \frac{1}{L_\Gamma(z)} \mathcal{J}_{\delta L_\Gamma}^{\overline{L_{\Gamma}}} (z) & \overline{\mathcal{J}_{K^{[l-2N_\Gamma+1]}}^{L_{\mathcal{C},*}}} (z) \end{vmatrix} \\ \tilde{H}_l &= \frac{L_{\mathcal{C},(-1)^{lN_{\mathcal{C}}}}}{L_{\Gamma,(-1)^{l+1N_\Gamma}}} H_{l-2N_\Gamma} \frac{\tilde{\tau}_{l+1}^{(2,1)}}{\tilde{\tau}_l^{(2,1)}} = \frac{L_{\mathcal{C},(-1)^{lN_{\mathcal{C}}}}}{L_{\Gamma,(-1)^{l+1n}}} H_{l-2N_\Gamma} \frac{\tilde{\tau}_{l+1}^{(1,2)}}{\tilde{\tau}_l^{(1,2)}}, \end{aligned}$$

$$\begin{aligned} \overline{\tilde{\phi}_{2,l}(z)} &= -\frac{\overline{L_{\Gamma,*}(z)}}{\overline{L_{\Gamma,(-1)^{l+1}N_{\Gamma}}}} \frac{H_{l-2N_{\Gamma}}}{\tilde{\tau}_l^{(2,1)}} \begin{vmatrix} \mathcal{J}_{C_{1,l-2N_{\Gamma}+1}}^{\overline{L_{\Gamma,*}}} - \mathcal{J}_{\frac{\Phi_{1,l-2N_{\Gamma}}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\Phi_{1,l-2N_{\Gamma}+1}}^{L_{\mathcal{C}}} \\ \vdots & \vdots \\ \mathcal{J}_{C_{1,l+2N_{\mathcal{C}}-1}}^{\overline{L_{\Gamma,*}}} - \mathcal{J}_{\frac{\Phi_{1,l+2N_{\mathcal{C}}-1}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* & \mathcal{J}_{\Phi_{1,l+2N_{\mathcal{C}}-1}}^{L_{\mathcal{C}}} \\ \mathcal{J}_{K_{C_1}^{[l-2N_{\Gamma}+1]}}^{\overline{L_{\Gamma,*}}}(\bar{z}) - \mathcal{J}_{\frac{K^{[l-2N_{\Gamma}+1]}}{L_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} \Xi \mathcal{L}_* + \frac{1}{L_{\Gamma,*}(z)} \mathcal{J}_{\delta L_{\Gamma,*}}^{\overline{L_{\Gamma,*}}}(\bar{z}) & \mathcal{J}_{K^{[l-2N_{\Gamma}+1]}}^{L_{\mathcal{C}}}(\bar{z}) \end{vmatrix} \\ &= \frac{\overline{L_{\mathcal{C},(-1)^{l+1}N_{\mathcal{C}}}}}{\overline{L_{\mathcal{C},*}(z)}} \frac{1}{\tilde{\tau}_l^{(1,2)}} \begin{vmatrix} \mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\Phi_{2,l-2N_{\Gamma}}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma,*}}} \Xi^{\dagger} \mathcal{L} & \mathcal{J}_{\Phi_{2,l-2N_{\Gamma}}}^{L_{\mathcal{C},*}} & \phi_{2,l-2N_{\Gamma}}(z) \\ \vdots & \vdots & \vdots \\ \mathcal{J}_{C_{2,l+2N_{\mathcal{C}}}}^{\overline{L_{\Gamma}}} - \mathcal{J}_{\frac{\Phi_{2,l+2N_{\mathcal{C}}}}{L_{\mathcal{C},*}}}^{\overline{L_{\Gamma,*}}} \Xi^{\dagger} \mathcal{L} & \mathcal{J}_{\Phi_{2,l+2N_{\mathcal{C}}}}^{L_{\mathcal{C},*}} & \phi_{2,l+2N_{\mathcal{C}}}(z) \end{vmatrix}. \end{aligned}$$

Let us further consider masses that are restricted to be supported on the diagonal [12]. For this discussion we need of

$$\mathcal{B}_{k,j}^{[i]} := (-1)^k \sum_{\substack{j_1+j_2+\dots+j_{k-j+1}=j \\ j_1+2j_2+\dots+(k-j+1)j_{k-j+1}=k}} \frac{k!}{j_1! \dots j_{k-j+1}!} (\zeta_i)^{-k-j},$$

$$\mathcal{B}^{[i]} := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathcal{B}_{1,1}^{[i]} & 0 & 0 & \dots & 0 \\ 0 & \mathcal{B}_{2,1}^{[i]} & \mathcal{B}_{2,2}^{[i]} & 0 & \dots & 0 \\ 0 & \mathcal{B}_{3,1}^{[i]} & \mathcal{B}_{3,2}^{[i]} & \mathcal{B}_{3,3}^{[i]} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{B}_{m_i-1,1}^{[i]} & \mathcal{B}_{m_i-1,2}^{[i]} & \mathcal{B}_{m_i-1,3}^{[i]} & \dots & \mathcal{B}_{m_i-1,m_i-1}^{[i]} \end{bmatrix}, \quad \Xi^i := \begin{bmatrix} \Xi_0^i & \frac{1}{1!1!} \Xi_1^i & \frac{1}{1!2!} \Xi_2^i & \dots & \frac{1}{1!(m_i-1)!} \Xi_{m_i-1}^i \\ \frac{1}{1!1!} \Xi_1^i & \frac{1}{1!1!} \Xi_2^i & \dots & \dots & 0 \\ \frac{1}{2!1!} \Xi_2^i & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ \frac{1}{(m_i-1)!1!} \Xi_{m_i-1}^i & 0 & \dots & \dots & 0 \end{bmatrix},$$

so that, if we choose $\Xi = \text{diag}(\Xi_1, \dots, \Xi_d)$ with $\Xi_i = \Xi^i \mathcal{B}^{[i]}$, the Geronimus–Uvarov perturbed sesquilinear form will be

$$\langle f(z), g(z) \rangle_{\bar{u}} = \left\langle \frac{L_{\mathcal{C}}(z)}{L_{\Gamma}(z)} u_z, f(z) \overline{g(z)} \right\rangle + \left\langle \sum_{i=1}^d \sum_{l=0}^{m_i-1} \Xi_l^i \frac{(-1)^l}{l!} \delta^{(l)}(z - \zeta_i), f(z) g_*(z) \right\rangle,$$

which is supported on the diagonal but, due to the mass terms, is not of zero order. The zero order appears when the higher derivatives of the Dirac functionals are cancelled

$$(31) \quad \langle f(z), g(z) \rangle_{\bar{u}} = \left\langle \frac{L_{\mathcal{C}}(z)}{L_{\Gamma}(z)} u_z, f(z) \overline{g(z)} \right\rangle + \left\langle \sum_{i=1}^d \Xi_0^i \delta(z - \zeta_i), f(z) g_*(z) \right\rangle,$$

which is achieved with $\Xi^i = \Xi_i = \begin{bmatrix} \xi^i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{m_i \times m_i}$ and, consequently, we will have the following

expression $\Xi \mathcal{L}_* = \Xi^{\dagger} \mathcal{L} = \text{diag} \left(\begin{bmatrix} 0 & \dots & 0 & \xi^1 \overline{L_{\Gamma,[1]}(\zeta_1)} \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \dots & 0 & \xi^d \overline{L_{\Gamma,[d]}(\zeta_d)} \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \right) =: C$, so that

$$\mathcal{J}_{\Phi}^L \Xi \mathcal{L}_* = \left[0, \dots, 0, \phi(\zeta_1) \xi^1 \overline{L_{\Gamma,[1]}(\zeta_1)}, \dots, 0, \dots, 0, \phi(\zeta_d) \xi^d \overline{L_{\Gamma,[d]}(\zeta_d)} \right].$$

As $L_{\mathcal{C}}(z)$ and $L_{\Gamma}(z)$ are coprime the reality of the first term in the RHS of (31) is ensured whenever $L_{\mathcal{C}}(z)$ and $L_{\Gamma}(z)$ are self-reciprocal polynomials, $L_{\mathcal{C},*}(z) = L_{\mathcal{C}}(z)$ and $L_{\Gamma,*}(z) = L_{\Gamma}(z)$. The mass term will be real whenever $\xi^i \in \mathbb{R}$, $i \in \{1, \dots, d\}$. If we are interested in the non-negative situation, then we must further impose that the self-reciprocal Laurent polynomials $L_{\mathcal{C}}(z)$ and $L_{\Gamma}(z)$ are such that $\frac{L_{\mathcal{C}}(z)}{L_{\Gamma}(z)} = \frac{|L_{\mathcal{C}}(z)|}{|L_{\Gamma}(z)|}$ and also that $\xi^i \geq 0$, $i \in \{1, \dots, d\}$.

Self-reciprocity for the Laurent polynomial $L = L_{2n} z^{-n} (z - \zeta_1)^{m_1} \cdots (z - \zeta_d)^{m_d}$, $m_1 + \cdots + m_d = 2n$, requires $\overline{L_{2n}}(\bar{\zeta}_1)^{m_1} \cdots (\bar{\zeta}_d)^{m_d} (z - (\bar{\zeta}_1)^{-1})^{m_1} \cdots (z - (\bar{\zeta}_d)^{-1})^{m_d} = L_{2n}(z - \zeta_1)^{m_1} \cdots (z - \zeta_d)^{m_d}$, that could be fulfilled if only if

$$L(z) = L_{2n} z^{-n} (z - \alpha_1)^{n_1} (z - (\bar{\alpha}_1)^{-1})^{n_1} \cdots (z - \alpha_r)^{n_r} (z - (\bar{\alpha}_r)^{-1})^{n_r} (z - \beta_1)^{2q_1} \cdots (z - \beta_s)^{2q_s},$$

where $n_1 + \cdots + n_r + q_1 + \cdots + q_s = n$ and $\alpha_i \notin \mathbb{T}$, $i \in \{1, \dots, r\}$ and $\beta_i \in \mathbb{T}$, $i \in \{1, \dots, s\}$, with

$$\arg L_{2n} = -\arg \alpha_1 - \cdots - \arg \alpha_r - \arg \beta_1 - \cdots - \arg \beta_s.$$

Therefore, for this real Toeplitz zero order diagonal case, and using

$$\tilde{\tau}_l := \begin{vmatrix} \mathcal{J}_{C_{l-2N_\Gamma}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l-2N_\Gamma}}{L_\mathfrak{e}}}^{L_\Gamma} C & \mathcal{J}_{\phi_{l-2N_\Gamma}}^{L_\mathfrak{e}} \\ \vdots & \vdots \\ \mathcal{J}_{C_{l+2N_\mathfrak{e}-1}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l+2N_\mathfrak{e}-1}}{L_\mathfrak{e}}}^{L_\Gamma} C & \mathcal{J}_{\phi_{l+2N_\mathfrak{e}-1}}^{L_\mathfrak{e}} \end{vmatrix},$$

we conclude that the perturbed orthogonal polynomials and its norms are

$$\begin{aligned} \tilde{\phi}_l(z) &= \frac{L_\mathfrak{e}, (-1)^{lN_\mathfrak{e}}}{L_\mathfrak{e}(z)} \frac{1}{\tilde{\tau}_l} \begin{vmatrix} \mathcal{J}_{C_{l-2N_\Gamma}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l-2N_\Gamma}}{L_\mathfrak{e}}}^{L_\Gamma} C & \mathcal{J}_{\phi_{l-2N_\Gamma}}^{L_\mathfrak{e}} & \phi_{l-2N_\Gamma}(z) \\ \vdots & \vdots & \vdots \\ \mathcal{J}_{C_{l+2N_\mathfrak{e}}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l+2N_\mathfrak{e}}}{L_\mathfrak{e}}}^{L_\Gamma} C & \mathcal{J}_{\phi_{l+2N_\mathfrak{e}}}^{L_\mathfrak{e}} & \phi_{l+2N_\mathfrak{e}}(z) \end{vmatrix} \\ &= -\frac{L_\Gamma(z)}{L_{\Gamma, (-1)^{lN_\Gamma}} \tilde{\tau}_l} \begin{vmatrix} \overline{\mathcal{J}_{C_{l-2N_\Gamma+1}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l-2N_\Gamma}}{L_\mathfrak{e}}}^{L_\Gamma} C} & \overline{\mathcal{J}_{\phi_{l-2N_\Gamma+1}}^{L_\mathfrak{e}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{l+2N_\mathfrak{e}-1}}^{\overline{L_\Gamma}} - \mathcal{J}_{\frac{\phi_{l+2N_\mathfrak{e}-1}}{L_\mathfrak{e}}}^{L_\Gamma} C} & \overline{\mathcal{J}_{\phi_{l+2N_\mathfrak{e}-1}}^{L_\mathfrak{e}}} \\ \mathcal{J}_{K_C^{[l-2N_\Gamma+1]}}^{\overline{L_\Gamma}}(z) - \mathcal{J}_{\frac{K^{[l-2N_\Gamma+1]}}{L_\mathfrak{e}}}^{\overline{L_\Gamma}} \bar{C} + \frac{1}{L_\Gamma(z)} \mathcal{J}_{\delta L_\Gamma}^{\overline{L_\Gamma}}(z) & \mathcal{J}_{K^{[l-2N_\Gamma+1]}}^{\overline{L_\mathfrak{e}}}(z) \end{vmatrix}, \\ \tilde{H}_l &= \frac{L_\mathfrak{e}, (-1)^{lN_\mathfrak{e}}}{L_{\Gamma, (-1)^{l+1N_\Gamma}} \tilde{\tau}_l} \frac{\tilde{\tau}_{l+1}}{\tilde{\tau}_l}. \end{aligned}$$

APPENDIX A. PROOFS

Proof of Proposition 10. From definition we have

$$\begin{aligned} \tilde{C}_1^{(1,2)}(z) &= \left\langle \tilde{\phi}_1^{(1,2)}(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(1,2)}}, & z \notin \overline{\text{supp}_2(\tilde{u}^{(1,2)})}, \\ \tilde{C}_1^{(2,1)}(z) &= \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}}, & z \notin \overline{\text{supp}_2(\tilde{u}^{(2,1)})}, \\ (\tilde{C}_2^{(1,2)}(z))^\dagger &= \left\langle \frac{1}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}}, & z \notin \overline{\text{supp}_1(\tilde{u}^{(1,2)})}, \\ (\tilde{C}_2^{(2,1)}(z))^\dagger &= \left\langle \frac{1}{\bar{z} - z_1}, (\tilde{\phi}_2^{(2,1)}(z_2))^\top \right\rangle_{\tilde{u}^{(2,1)}}, & z \notin \overline{\text{supp}_1(\tilde{u}^{(2,1)})}. \end{aligned}$$

Then, (11) is proven as follows

$$\begin{aligned}
(C_2(z))^\dagger (\Omega_2^{(1,2)})^\dagger - L_\Gamma^{(1)}(\bar{z}) (\tilde{C}_2^{(1,2)}(z))^\dagger &= \left\langle \frac{1}{\bar{z} - z_1}, (\phi_2(z_2))^\top \right\rangle_u (\Omega_2^{(1,2)})^\dagger - L_\Gamma^{(1)}(\bar{z}) \left\langle \frac{1}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}} \\
&= \left\langle \frac{1}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top L_\epsilon^{(2)}(z_2) \right\rangle_u - \left\langle \frac{L_\Gamma^{(1)}(\bar{z})}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}} \\
&= \left\langle \frac{L_\Gamma^{(1)}(z_1)}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}} - \left\langle \frac{L_\Gamma^{(1)}(\bar{z})}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}} \\
&= - \left\langle \frac{L_\Gamma^{(1)}(\bar{z}) - L_\Gamma^{(1)}(z_1)}{\bar{z} - z_1}, (\tilde{\phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\tilde{u}^{(1,2)}}.
\end{aligned}$$

For (12) we have

$$\begin{aligned}
\Omega_1^{(2,1)} C_1(z) - \tilde{C}_1^{(2,1)}(z) \overline{L_\Gamma^{(2)}}(z) &= \Omega_1^{(2,1)} \left\langle \phi_1(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_u - \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}} \overline{L_\Gamma^{(2)}}(z) \\
&= \left\langle L_\epsilon^{(1)}(z_1) \tilde{\phi}_1^{(2,1)}(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_u - \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(\bar{z})}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}} \\
&= \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(z_2)}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}} - \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(\bar{z})}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}} \\
&= - \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(\bar{z}) - L_\Gamma^{(2)}(z_2)}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(2,1)}}.
\end{aligned}$$

The formula (13) follows from $\Omega_1^{(1,2)} \tilde{C}_1^{(1,2)}(z) = \left\langle L_\Gamma^{(1)}(z_1) \phi_1(z_1), \frac{1}{\bar{z} - z_2} \right\rangle_{\tilde{u}^{(1,2)}} = \left\langle \phi_1(z_1), \frac{L_\epsilon^{(2)}(z_2)}{\bar{z} - z_2} \right\rangle_u$. For (14) we notice

$$\begin{aligned}
(\tilde{C}_2^{(2,1)}(z))^\dagger (\Omega_2^{(2,1)})^\dagger &= \left\langle \frac{1}{\bar{z} - z_1}, (\tilde{\phi}_2(z_2))^\top \right\rangle_{\tilde{u}^{(2,1)}} (\Omega_2^{(2,1)})^\dagger \\
&= \left\langle \frac{1}{\bar{z} - z_1}, L_\Gamma^{(2)}(z_2) (\phi_2(z_2))^\top \right\rangle_{\tilde{u}^{(2,1)}} \\
&= \left\langle \frac{L_\epsilon^{(1)}(z_1)}{\bar{z} - z_1}, (\phi_2(z_2))^\top \right\rangle_u.
\end{aligned}$$

□

Proof of Proposition 12. The relations

$$(\phi_2(z_1))^\dagger (\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} = \overline{L_\epsilon^{(2)}(z_1)} (\tilde{\phi}_2^{(1,2)}(z_1))^\dagger (\tilde{H}^{(1,2)})^{-1}, \quad (\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \tilde{\phi}_1^{(1)}(z_2) = L_\Gamma^{(1)}(z_2) H^{-1} \phi_1(z_2),$$

imply

$$\begin{aligned}
&\left[(\phi_2(z_1))^\dagger \right]^{[l]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[l]} \left[\tilde{\phi}_1^{(1,2)}(z_2) \right]^{[l]} + \left[(\phi_2(z_1))^\dagger \right]^{\geq [l]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{\geq [l, l]} \left[\tilde{\phi}_1^{(1)}(z_2) \right]^{[l]} \\
&\quad = \overline{L_\epsilon^{(2)}(z_1)} \left[(\tilde{\phi}_2^{(1,2)}(z_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(1,2)})^{-1} \right]^{[l]} \left[\tilde{\phi}_1^{(1,2)}(z_2) \right]^{[l]}, \\
&\left[(\phi_2(z_1))^\dagger \right]^{[l]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[l]} \left[\tilde{\phi}_1^{(1,2)}(z_2) \right]^{[l]} + \left[(\phi_2(z_1))^\dagger \right]^{[l]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[l, \geq l]} \left[\tilde{\phi}_1^{(1,2)}(z_2) \right]^{\geq [l]} \\
&\quad = L_\Gamma^{(1)}(z_2) \left[(\phi_2(z_1))^\dagger \right]^{[l]} \left[H^{-1} \right]^{[l]} \left[\phi_1(z_2) \right]^{[l]},
\end{aligned}$$

so that

$$\begin{aligned} & \left[(\tilde{\Phi}_1^{(1)}(z_2))^\top \right]^{[1]} \left[(\tilde{H}^{(1,2)})^{-1} \tilde{\Omega}_2^{(1,2)} \right]^{[l, \geq 1]} \left[\overline{\Phi_2(z_1)} \right]^{[\geq 1]} - \left[(\tilde{\Phi}_1^{(1,2)}(z_2))^\top \right]^{[\geq 1]} \left[(\tilde{H}^{(1,2)})^{-1} \tilde{\Omega}_2^{(1,2)} \right]^{[\geq 1, l]} \left[\overline{\Phi_2(z_1)} \right]^{[1]} \\ & = \overline{L_{\mathcal{C}}^{(2)}(z_1)} \tilde{K}^{(1,2), [1]}(\bar{z}_1, z_2) - L_{\Gamma}^{(1)}(z_2) K^{[1]}(\bar{z}_1, z_2), \end{aligned}$$

and (17) follows. On the other hand, we have

$$(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \Phi_1(z_2) = L_{\mathcal{C}}^{(1)}(z_2) (\tilde{H}^{(2,1)})^{-1} \tilde{\Phi}_1^{(2,1)}(z_2), \quad (\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger (\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} = \overline{L_{\Gamma}^{(2)}(z_1)} (\Phi_2(z_1))^\dagger H^{-1}.$$

Therefore,

$$\begin{aligned} & \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1]} [\Phi_1(z_2)]^{[1]} + \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l, \geq 1]} [\Phi_1(z_2)]^{[\geq 1]} \\ & = L_{\mathcal{C}}^{(1)}(z_2) \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \right]^{[1]} [\tilde{\Phi}_1^{(2,1)}(z_2)]^{[1]}, \\ & \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1]} [\Phi_1(z_2)]^{[1]} + \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[\geq 1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[\geq 1, l]} [\Phi_1(z_2)]^{[1]} \\ & = \overline{L_{\Gamma}^{(2)}(z_1)} \left[(\Phi_2(z_1))^\dagger \right]^{[1]} [H^{-1}]^{[1]} [\Phi_1(z_2)]^{[1]}, \end{aligned}$$

from where we deduce that

$$\begin{aligned} & \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l, \geq 1]} [\Phi_1(z_2)]^{[\geq 1]} - \left[(\tilde{\Phi}_2^{(2,1)}(z_1))^\dagger \right]^{[\geq 1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[\geq 1, l]} [\Phi_1(z_2)]^{[1]} \\ & = L_{\mathcal{C}}^{(1)}(z_2) \tilde{K}^{(2,1), [1]}(\bar{z}_1, z_2) - \overline{L_{\Gamma}^{(2)}(z_1)} K^{[1]}(\bar{z}_1, z_2), \end{aligned}$$

and (18) follows. \square

Proof of Proposition 13. The following equations

$$(C_2(x_1))^\dagger (\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} - L_{\Gamma}^{(1)}(\bar{x}_1) (\tilde{C}_2^{(1,2)}(x_1))^\dagger (\tilde{H}^{(1,2)})^{-1} = - \left\langle \delta L_{\Gamma}^{(1)}(\bar{x}_1, z_1), (\tilde{\Phi}_2^{(1,2)}(z_2))^\top \right\rangle_{\bar{u}(1)} (\tilde{H}^{(1,2)})^{-1},$$

$$(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \tilde{\Phi}_1^{(1,2)}(x_2) = L_{\Gamma}^{(1)}(x_2) H^{-1} \Phi_1(x_2),$$

imply

$$\begin{aligned} & \left[(C_2(x_1))^\dagger \right]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[1]} + \left[(C_2(x_1))^\dagger \right]^{[\geq 1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[\geq 1, l]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[1]} \\ & - L_{\Gamma}^{(1)}(\bar{x}_1) \left[(\tilde{C}_2^{(1,2)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[1]} \\ & = - \left\langle \delta L_{\Gamma}^{(1)}(\bar{x}_1, z_1), \left(\left[\tilde{\Phi}_2^{(1,2)}(z_2) \right]^{[1]} \right)^\top \right\rangle_{\bar{u}(1,2)} \left[(\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[1]}, \\ & [C_2(x_1)]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[1]} + [C_2(x_1)]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[l, \geq 1]} [\tilde{\Phi}_1^{(1,2)}(x_2)]^{[\geq 1]} \\ & = L_{\Gamma}^{(1)}(x_2) [C_2(x_1)]^{[1]} [H^{-1}]^{[1]} [\Phi_1(x_2)]^{[1]}, \end{aligned}$$

and, consequently,

$$\begin{aligned} & \left[(\tilde{\Phi}_1^{(1,2)}(x_2))^\top \right]^{[\geq 1]} \left[(\tilde{H}^{(1,2)})^{-1} (\tilde{\Omega}_2^{(1,2)}) \right]^{[\geq 1, l]} \left[\overline{C_2(x_1)} \right]^{[1]} + L_{\Gamma}^{(1)}(\bar{x}_1) \tilde{K}_{C_2}^{(1,2), [1]}(\bar{x}_1, x_2) \\ & = \left[(\tilde{\Phi}_1^{(1,2)}(x_2))^\top \right]^{[1]} \left[(\tilde{H}^{(1,2)})^{-1} (\tilde{\Omega}_2^{(1,2)}) \right]^{[l, \geq 1]} \left[\overline{C_2(x_1)} \right]^{[\geq 1]} + \left\langle \delta L_{\Gamma}^{(1)}(\bar{x}_1, z_1), \overline{\tilde{K}^{(1,2), [1]}(\bar{z}_2, x_2)} \right\rangle_{\bar{u}(1)} \\ & \quad + L_{\Gamma}^{(1)}(x_2) K_{C_2}^{[1]}(\bar{x}_1, x_2). \end{aligned}$$

and (19) follows. Now, to check (20), observe that

$$\begin{aligned} (\tilde{C}_2^{(2,1)}(x_1))^\dagger (\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} &= \left\langle \frac{L_{\mathfrak{e}}^{(1)}(z_1)}{\bar{x}_1 - z_1}, (\phi_2(z_2))^\top \right\rangle_{\mathfrak{u}} H^{-1}, \\ (\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \phi_1(x_2) &= L_{\mathfrak{e}}^{(1)}(x_2) (\tilde{H}^{(2,1)})^{-1} \tilde{\phi}_1^{(2,1)}(x_2). \end{aligned}$$

Thus,

$$\begin{aligned} & \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1]} [\phi_1(x_2)]^{[1]} + \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[\geq 1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[\geq 1, 1]} [\phi_1(x_2)]^{[1]} \\ &= \left\langle \frac{L_{\mathfrak{e}}^{(1)}(z_1)}{\bar{x}_1 - z_1}, [(\phi_2(z_2))^\top]^{[1]} [H^{-1}]^{[1]} \right\rangle_{\mathfrak{u}} [\phi_1(x_2)]^{[1]}, \\ & \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1]} [\phi_1(x_2)]^{[1]} + \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1, \geq 1]} [\phi_1(x_2)]^{[\geq 1]} \\ &= L_{\mathfrak{e}}^{(1)}(x_2) \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \right]^{[1]} [\tilde{\phi}_1^{(2,1)}(x_2)]^{[1]}, \end{aligned}$$

and we deduce

$$\begin{aligned} & \left\langle \frac{L_{\mathfrak{e}}^{(1)}(z_1)}{\bar{x}_1 - z_1}, \overline{K^{[1]}(\bar{z}_2, x_2)} \right\rangle_{\mathfrak{u}} - \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[\geq 1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2)} \right]^{[\geq 1, 1]} [\phi_1(x_2)]^{[1]} = L_{\mathfrak{e}}^{(1)}(x_2) \tilde{K}_{C_2}^{(2), [1]}(\bar{x}_1, x_2) \\ & - \left[(\tilde{C}_2^{(2,1)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[1, \geq 1]} [\phi_1(x_2)]^{[\geq 1]}. \end{aligned}$$

To get (21) we notice that

$$\begin{aligned} (\phi_2(x_1))^\dagger (\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} &= \overline{L_{\mathfrak{e}}^{(2)}(x_1)} (\tilde{\phi}_2^{(1,2)}(x_1))^\dagger (\tilde{H}^{(1,2)})^{-1}, \\ (\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \tilde{C}_1^{(1,2)}(x_2) &= H^{-1} \left\langle \phi_1(z_1), \frac{L_{\mathfrak{e}}^{(2)}(z_2)}{\bar{x}_2 - z_2} \right\rangle_{\mathfrak{u}}, \end{aligned}$$

and, therefore,

$$\begin{aligned} & \left[(\phi_2(x_1))^\dagger \right]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{C}_1^{(1,2)}(x_2)]^{[1]} + \left[(\phi_2(x_1))^\dagger \right]^{[\geq 1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[\geq 1, 1]} [\tilde{C}_1^{(1,2)}(z)]^{[1]}, \\ &= \overline{L_{\mathfrak{e}}^{(2)}(x_1)} \left[(\tilde{\phi}_2^{(1,2)}(x_1))^\dagger \right]^{[1]} \left[(\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{C}_1^{(1,2)}(x_2)]^{[1]} \\ & \left[(\phi_2(x_1))^\dagger \right]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[1]} [\tilde{C}_1^{(1,2)}(x_2)]^{[1]} + \left[(\phi_2(x_1))^\dagger \right]^{[1]} \left[(\Omega_2^{(1,2)})^\dagger (\tilde{H}^{(1,2)})^{-1} \right]^{[1, \geq 1]} [\tilde{C}_1^{(1,2)}(x_2)]^{[\geq 1]} \\ &= \left[(\phi_2(x_1))^\dagger \right]^{[1]} \left\langle [H^{-1}]^{[1]} [\phi_1(z_1)]^{[1]}, \frac{L_{\mathfrak{e}}^{(2)}(z_2)}{\bar{x}_2 - z_2} \right\rangle_{\mathfrak{u}}, \end{aligned}$$

so that

$$\begin{aligned} & \left[(\tilde{C}_1^{(1,2)}(x_2))^\top \right]^{[1]} \left[(\tilde{H}^{(1,2)})^{-1} \bar{\Omega}_2^{(1,2)} \right]^{[1, \geq 1]} \left[\overline{(\phi_2(x_1))} \right]^{[\geq 1]} - \left[(\tilde{C}_1^{(1,2)}(x_2))^\top \right]^{[\geq 1]} \left[(\tilde{H}^{(1,2)})^{-1} \bar{\Omega}_2^{(1,2)} \right]^{[\geq 1, 1]} \left[\overline{(\phi_2(x_1))} \right]^{[1]} \\ &= \overline{L_{\mathfrak{e}}^{(2)}(x_1)} \tilde{K}_{C_1}^{(1,2), [1]}(\bar{x}_1, x_2) - \left\langle K^{[1]}(\bar{x}_1, z_1), \frac{L_{\mathfrak{e}}^{(2)}(z_2)}{\bar{x}_2 - z_2} \right\rangle_{\mathfrak{u}}. \end{aligned}$$

Finally, we prove (22). For that aim consider

$$\begin{aligned} & (\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} C_1(x_2) - \overline{L_{\Gamma}^{(2)}(x_2)} (\tilde{H}^{(2,1)})^{-1} \tilde{C}_1^{(2,1)}(x_2) = -(\tilde{H}^{(2,1)})^{-1} \left\langle \tilde{\phi}_1^{(2,1)}(z_1), \delta L_{\Gamma}^2(\bar{x}_2, z_2) \right\rangle_{\bar{\mathfrak{u}}^{(2,1)}}, \\ & (\tilde{\phi}_2^{(2,1)}(x_1))^\dagger (\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} = \overline{L_{\Gamma}^{(2)}(x_1)} (\phi_2(x_1))^\dagger H^{-1}, \end{aligned}$$

and as a consequence we find

$$\begin{aligned}
& \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l]} [C_1(x_2)]^{[l]} + \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l, \geq l]} [C_1(x_2)]^{[\geq l]} \\
& \quad - \overline{L_\Gamma^{(2)}}(x_2) \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(2,1)})^{-1} \right]^{[l]} \left[\tilde{C}_1^{(2)}(x_2) \right]^{[l]} \\
& \quad = - \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left\langle \left[(\tilde{H}^{(2,1)})^{-1} \right]^{[l]} \left[\tilde{\Phi}_1^{(2,1)}(z_1) \right]^{[l]}, \delta L_\Gamma^{(2)}(\bar{x}_2, z_2) \right\rangle_{\bar{u}^{(2,1)}}, \\
& \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l]} [C_1(x_2)]^{[l]} + \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[\geq l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[\geq l, l]} [C_1(x_2)]^{[l]} \\
& \quad = \overline{L_\Gamma^{(2)}}(x_1) \left[(\Phi_2(x_1))^\dagger \right] [H^{-1}]^{[l]} [C_1(x_2)]^{[l]},
\end{aligned}$$

that gives

$$\begin{aligned}
& \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[\geq l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[\geq l, l]} [C_1(x_2)]^{[l]} - \left[(\tilde{\Phi}_2^{(2,1)}(x_1))^\dagger \right]^{[l]} \left[(\tilde{H}^{(2,1)})^{-1} \Omega_1^{(2,1)} \right]^{[l, \geq l]} [C_1(x_2)]^{[\geq l]} \\
& \quad = \overline{L_\Gamma^{(2)}}(x_1) K_{C_1}^{[l]}(\bar{x}_1, x_2) - \overline{L_\Gamma^{(2)}}(x_2) \tilde{K}_{C_1}^{(2,1), [l]}(\bar{x}_1, x_2) + \left\langle \tilde{K}^{(2,1), [l]}(\bar{x}_1, z_1), \delta L_\Gamma^{(2)}(\bar{x}_2, z_2) \right\rangle_{\bar{u}^{(2,1)}}.
\end{aligned}$$

□

Proof of Proposition 14. First, let us show that, for $l = 0, \dots, m_j^{(2)} - 1$ we have

$$\left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_j^{(2)}} \left\langle \tilde{\Phi}_{1,k}^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(\bar{z})}{\bar{z} - z_2} \right\rangle_{\frac{L_\Gamma^{(1)}(z_1)}{(L_\Gamma^{(2)}(z_2))} u} = 0.$$

This is a consequence of

$$\begin{aligned}
& \left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_j^{(2)}} \left\langle \tilde{\Phi}_{1,k}^{(2,1)}(z_1), \frac{L_\Gamma^{(2)}(\bar{z})}{\bar{z} - z_2} \right\rangle_{\frac{L_\Gamma^{(1)}(z_1)}{(L_\Gamma^{(2)}(z_2))} u} = \left\langle u_{z_1, \bar{z}_2}, L_\Gamma^{(1)}(z_1) \tilde{\Phi}_{1,k}^{(2,1)}(z_1) \otimes (\overline{L_\Gamma^{(2)}(z_2)})^{-1} \left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_j^{(2)}} \frac{\overline{L_\Gamma^{(2)}(z)}}{z - \bar{z}_2} \right\rangle \\
& \quad = \sum_{v=0}^l \binom{l}{v} \left\langle u_{z_1, \bar{z}_2}, L_\Gamma^{(1)}(z_1) \tilde{\Phi}_{1,k}^{(2,1)}(z_1) \otimes (\overline{L_\Gamma^{(2)}(z_2)})^{-1} \left(\left. \frac{d^v \overline{L_\Gamma^{(2)}(z)}}{dz^v} \right|_{z=\bar{\zeta}_j^{(2)}} \left. \frac{d^{l-v}}{dz^{l-v}} \right|_{z=\bar{\zeta}_j^{(2)}} \frac{1}{z - \bar{z}_2} \right) \right\rangle,
\end{aligned}$$

but $\left. \frac{d^v \overline{L_\Gamma^{(2)}(z)}}{dz^v} \right|_{z=\bar{\zeta}_j^{(2)}} = 0$ for $v \in \{0, \dots, m_j^{(2)} - 1\}$, and since $\text{supp}_2(u) \cap \overline{\sigma(L_\Gamma^{(2)})} = \emptyset$, we get (23). Thus,

$$\left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_j^{(2)}} \overline{L_\Gamma^{(2)}}(z) \tilde{C}_{1,k}^{(2,1)}(z) = \sum_{i=1}^{d^{(2)}} \left\langle L_\Gamma^{(1)} \xi_i^{(2)}, \tilde{\Phi}_{1,k}^{(2,1)} \right\rangle \left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_j^{(2)}} \overline{L_{\Gamma, [i]}^{(2)}}(z) \begin{bmatrix} (z - \bar{\zeta}_i^{(2)})^{m_i^{(2)} - 1} \\ (z - \bar{\zeta}_i^{(2)})^{m_i^{(2)} - 2} \\ \vdots \\ 1 \end{bmatrix}.$$

For $l > 0$, we have

$$\begin{aligned}
& \left. \frac{d^l}{dz^l} \right|_{z=\bar{\zeta}_{2,j}} \overline{L_{\Gamma, [i]}^{(2)}}(z) (z - \bar{\zeta}_i^{(2)})^m = \sum_{v=0}^m \binom{l}{m-v} \left(\left. \frac{d^{l-m+v}}{dz^{l-m+v}} \right|_{z=\bar{\zeta}_j^{(2)}} \overline{L_{\Gamma, [i]}^{(2)}}(z) \right) \frac{m!}{(m-v)!} (\bar{\zeta}_j^{(2)} - \bar{\zeta}_i^{(2)})^v \\
& \quad = \sum_{v=0}^m \sum_{\sigma=1}^{l-m+v} \frac{l! m!}{\sigma! (m-v)!} \overline{(L_{\Gamma, [i]}^{(2)})^{(\sigma)}(\zeta_j^{(2)})} (\bar{\zeta}_j^{(2)} - \bar{\zeta}_i^{(2)})^v.
\end{aligned}$$

But, if $i \neq j$, $(L_{\Gamma, [i]}^{(2)})^{(\sigma)}(\zeta_j^{(2)}) = 0$ for $\sigma \in \{0, 1, \dots, m_j^{(2)} - 1\}$, which is our case because $l \in \{0, 1, \dots, m_j^{(2)} - 1\}$; when $i = j$ we get that only terms with $v = 0$ will survive and, therefore, $m \leq l$ with

$$\frac{1}{l!} \frac{d^l}{dz^l} \Big|_{z=\bar{\zeta}_j^{(2)}} \overline{L_{\Gamma, [j]}^{(2)}}(z)(z - \bar{\zeta}_j^{(2)})^m = \ell_{[j], l-m}^{(2)}.$$

To show (24) let's compute $\frac{1}{l!} \frac{d^l}{dz^l} \Big|_{z=\zeta_j^{(1)}} \overline{L_{\Gamma}^{(1)}(z) \tilde{C}_{2,k}^{(1,2)}(z)}$ with $l \in \{0, 1, \dots, m_j^{(1)} - 1\}$. For that aim we evaluate

$$\begin{aligned} \frac{d^l}{dz^l} \Big|_{z=\zeta_j^{(1)}} \left\langle \frac{L_{\Gamma}^{(1)}(\bar{z})}{\bar{z} - z_1}, \tilde{\Phi}_{2,k}^{(1,2)}(z_2) \right\rangle_{\frac{(\overline{L_{\mathfrak{C}}^{(1)}(z_2))}}{L_{\Gamma}^{(1)}(z_1)} u} &= \left\langle \frac{d^l}{dz^l} \Big|_{z=\zeta_j^{(1)}} \frac{L_{\Gamma}^{(1)}(\bar{z})}{\bar{z} - z_1}, \tilde{\Phi}_{2,k}^{(1,2)}(z_2) \right\rangle_{\frac{(\overline{L_{\mathfrak{C}}^{(1)}(z_2))}}{L_{\Gamma}^{(1)}(z_1)} u} \\ &= \sum_{k=0}^l \binom{l}{k} (l-k)! \frac{d^k L_{\Gamma}^{(1)}(\bar{z})}{dz^k} \Big|_{z=\zeta_j^{(1)}} (-1)^{l-k} \left\langle \frac{1}{(\zeta_j^{(1)} - z_1)^{l-k+1}}, \tilde{\Phi}_{2,k}^{(1,2)}(z_2) \right\rangle_{\frac{(\overline{L_{\mathfrak{C}}^{(1)}(z_2))}}{L_{\Gamma}^{(1)}(z_1)} u}, \end{aligned}$$

that, remembering that the zeros are not in support of the linear functional, vanishes. Finally we realize, that

$$\begin{aligned} \frac{1}{l!} \frac{d^l}{dz^l} \Big|_{z=\zeta_j^{(1)}} L_{\Gamma, [i]}^{(1)}(\bar{z})(\bar{z} - \zeta_i^{(1)})^m &= \sum_{k=0}^l \binom{l}{k} \frac{d^k L_{\Gamma, [i]}^{(1)}(\bar{z})}{dz^k} \Big|_{z=\zeta_j^{(1)}} \frac{m!}{(l-k)!} (\zeta_j^{(1)} - \zeta_i^{(1)})^{m-l+k} \\ &= \begin{cases} 0, & i \neq j, \\ \bar{\ell}_{[j], l-m}^{(1)}, & i = j. \end{cases} \end{aligned}$$

□

Proof of Theorem 1. From (16) we get

$$\mathcal{J}_{L_{\Gamma}^{(2)} \tilde{C}_{1,l}^{(2,1)}} = [(\Omega_1^{(2,1)})_{l, l-2N_{\Gamma}}, \dots, (\Omega_1^{(2,1)})_{l, l+2r-1}] \begin{bmatrix} \overline{\mathcal{J}_{C_{1, l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{1, l+2N_{\mathfrak{C}}-1}}^{L_{\Gamma}^{(2)}}} \end{bmatrix} + (\Omega_1^{(2,1)})_{l, l+2N_{\mathfrak{C}}} \overline{\mathcal{J}_{C_{1, l+2N_{\mathfrak{C}}}^{(2)}}^{L_{\Gamma}^{(2)}}}.$$

Using (23) and

$$(32) \quad (\Omega_1^{(2)})_{l, l-2N_{\Gamma}} \phi_{1, l-2N_{\Gamma}}(z) + \dots + (\Omega_1^{(2)})_{l, l+2N_{\mathfrak{C}}} \phi_{1, l+2N_{\mathfrak{C}}}(z) = L_{\mathfrak{C}}^{(1)}(z) \tilde{\Phi}_1^{(2,1)}(z),$$

see Proposition 8, we deduce that

$$\mathcal{J}_{L_{\Gamma}^{(2)} \tilde{C}_{1,l}^{(2,1)}} = [(\Omega_1^{(2,1)})_{l, l-2N_{\Gamma}}, \dots, (\Omega_1^{(2,1)})_{l, l+2N_{\mathfrak{C}}-1}] \begin{bmatrix} \langle \xi^{(2)}, \phi_{1, l-2N_{\Gamma}} \rangle \\ \vdots \\ \langle \xi^{(2)}, \phi_{1, l+2N_{\mathfrak{C}}-1} \rangle \end{bmatrix} \mathcal{L}^{(2)} + (\Omega_1^{(2,1)})_{l, l+2N_{\mathfrak{C}}} \langle \xi^{(2)}, \phi_{1, l+2N_{\mathfrak{C}}} \rangle \mathcal{L}^{(2)},$$

and, consequently,

$$\begin{aligned} & [(\Omega_1^{(2,1)})_{l, l-2N_{\Gamma}}, \dots, (\Omega_1^{(2,1)})_{l, l+2N_{\mathfrak{C}}-1}] \begin{bmatrix} \overline{\mathcal{J}_{C_{1, l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1, l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)} \\ \vdots \\ \overline{\mathcal{J}_{C_{1, l+2N_{\mathfrak{C}}-1}}^{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1, l+2N_{\mathfrak{C}}-1} \rangle \mathcal{L}^{(2)} \end{bmatrix} \\ &= -(\Omega_1^{(2,1)})_{l, l+2N_{\mathfrak{C}}} \left(\overline{\mathcal{J}_{C_{1, l+2N_{\mathfrak{C}}}^{(2)}}^{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1, l+2N_{\mathfrak{C}}} \rangle \mathcal{L}^{(2)} \right). \end{aligned}$$

Taking spectral jets along $L_{\mathfrak{e}}^{(1)}(z)$ of (32) we get

$$[(\Omega_1^{(2,1)})_{l,l-2N_\Gamma}, \dots, (\Omega_1^{(2,1)})_{l,l+2N_{\mathfrak{e}}-1}] \begin{bmatrix} \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} \\ \vdots \\ \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(1)}} \end{bmatrix} = -(\Omega_1^{(2,1)})_{l,l+2N_{\mathfrak{e}}} \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}}.$$

The previous two relations leads to

$$\begin{aligned} & [(\Omega_1^{(2,1)})_{l,l-2N_\Gamma}, \dots, (\Omega_1^{(2,1)})_{l,l+2N_{\mathfrak{e}}-1}] \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_\Gamma}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(1)}} \end{bmatrix} \\ &= -(\Omega_1^{(2,1)})_{l,l+2N_{\mathfrak{e}}} \left[\overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)}, \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}} \right]. \end{aligned}$$

and we have

$$\begin{aligned} & [(\Omega_1^{(2,1)})_{l,l-2N_\Gamma}, \dots, (\Omega_1^{(2,1)})_{l,l+2N_{\mathfrak{e}}-1}] \\ &= -L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)} \left[\overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)}, \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_\Gamma}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(1)}} \end{bmatrix}^{-1}. \end{aligned}$$

Recalling (32) we conclude with the proof of this Christoffel formula. From this equation, we also obtain

$$\begin{aligned} & (\Omega_1^{(2,1)})_{l,l-2N_\Gamma} \\ &= -L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)} \left[\overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)}, \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_\Gamma}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(1)}} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

and recall (9). Thus, in terms of last quasideterminants [26, 36, 12] we find

$$\begin{aligned} \tilde{\Phi}_{1,l}^{(2,1)}(z) &= \frac{L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)}}{L_{\mathfrak{e}}^{(1)}(z)} \Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_\Gamma}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} & \Phi_{1,l-2N_\Gamma}(z) \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}} & \Phi_{1,l+2N_{\mathfrak{e}}}(z) \end{bmatrix}, \\ \tilde{H}_l^{(2,1)} &= \frac{L_{\mathfrak{e},(-1)^{lN_{\mathfrak{e}}}}^{(1)}}{L_{\Gamma,(-1)^{lN_\Gamma}}^{(2)}} H_{l-2N_\Gamma} \Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_\Gamma}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(1)}} & 1 \\ \overline{\mathcal{J}_{C_{1,l-2N_\Gamma+1}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_\Gamma+1} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l-2N_\Gamma+1}}^{L_{\mathfrak{e}}^{(1)}} & 0 \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(2)} & \mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(1)}} & 0 \end{bmatrix}, \end{aligned}$$

that, expressing the quasideterminant as a quotient of determinants, gives (25) and (26).

On the one hand we observe that (15) implies

$$(33) \quad \left[(\Omega_2^{(1,2)})_{l,l-2N_\Gamma}, \dots, (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}-1} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma}}^{L_\Gamma^{(1)}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(1)}}} \end{bmatrix} + (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}} \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(1)}}} = \overline{\mathcal{J}_{\tilde{C}_{2,l}}^{L_\Gamma^{(1)}}}.$$

On the other hand, from Proposition 8 we know that

$$(34) \quad (\Omega_2^{(1,2)})_{l,l-2N_\Gamma} \phi_{2,l-2N_\Gamma}(z) + \dots + (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}-1} \phi_{2,l+2N_{\mathfrak{e}}-1}(z) + (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}} \phi_{2,l+2N_{\mathfrak{e}}}(z) = L_{\mathfrak{e}}^{(2)}(z) \tilde{\phi}_{2,l}^{(1,2)}(z),$$

and taking into account (24) we conclude

$$(35) \quad \left[(\Omega_2^{(1,2)})_{l,l-2N_\Gamma}, \dots, (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}-1} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_\Gamma} \rangle \mathcal{L}_\Gamma^{(1)} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}_\Gamma^{(1)} \end{bmatrix} \\ = -(\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}} \left(\overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}_\Gamma^{(1)} \right).$$

Now, the computation of the spectral jet of (34) along $L_{\mathfrak{e}}^{(2)}(z)$ leads to

$$\left[(\Omega_2^{(1,2)})_{l,l-2N_\Gamma}, \dots, (\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}-1} \right] \begin{bmatrix} \mathcal{J}_{\phi_{2,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(2)}} \\ \vdots \\ \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(2)}} \end{bmatrix} = -(\Omega_2^{(1,2)})_{l,l+2N_{\mathfrak{e}}} \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(2)}}.$$

Hence, if we gather all this information together, we obtain

$$\begin{aligned} & \left[(\Omega_1^{(1,2)})_{l,l-2N_\Gamma}, \dots, (\Omega_1^{(1,2)})_{l,l+2N_{\mathfrak{e}}-1} \right] \\ &= -L_{\mathfrak{e},(-1)^{1N_{\mathfrak{e}}}}^{(2)} \left[\overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(1)}, \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(2)}} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_\Gamma} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(2)}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(2)}} \end{bmatrix}^{-1}, \end{aligned}$$

which as a byproduct offers

$$(36) \quad (\Omega_2^{(1,2)})_{l,l-2N_\Gamma} \\ = -L_{\mathfrak{e},(-1)^{1N_{\mathfrak{e}}}}^{(2)} \left[\overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(1)}, \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(2)}} \right] \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_\Gamma} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l-2N_\Gamma}}^{L_{\mathfrak{e}}^{(2)}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{L_\Gamma^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}-1}}^{L_{\mathfrak{e}}^{(2)}} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, we have proven the following last quasideterminantal expressions

$$\tilde{\Phi}_{2,l}^{(1,2)}(z) = \frac{L_{\mathfrak{e},(-1)^{1N_{\mathfrak{e}}}}^{(2)}}{L_{\mathfrak{e}}^{(2)}(z)} \Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{L_{\mathfrak{e}}^{(2)}} & \phi_{2,l-2N_{\Gamma}}(z) \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(2)}} & \phi_{2,l+2N_{\mathfrak{e}}}(z) \end{bmatrix},$$

$$\bar{\tilde{H}}_l^{(1,2)} = \frac{L_{\mathfrak{e},(-1)^{1N_{\mathfrak{e}}}}^{(2)}}{L_{\Gamma,(-1)^{1n}}^{(2)}} \bar{H}_{l-2N_{\Gamma}} \Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{L_{\mathfrak{e}}^{(2)}} & 1 \\ \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}+1} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l-2N_{\Gamma}+1}}^{L_{\mathfrak{e}}^{(2)}} & 0 \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}}}^{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}} \rangle \mathcal{L}^{(1)} & \mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}}}^{L_{\mathfrak{e}}^{(2)}} & 0 \end{bmatrix},$$

from where the determinantal formulas (28) and (29) follow immediately.

We will prove (30) y (27) simultaneously. Let's write (19) and (22) as follows

$$\sum_{k=0}^{l-1} \overline{L_{\Gamma}^{(1)}(x_1) \tilde{C}_{2,k}^{(1,2)}(x_1) (\tilde{H}_k^{(1,2)})^{-1} \tilde{\Phi}_{1,k}^{(1,2)}(x_2) - L_{\Gamma}^{(1)}(x_2) K_{C_2}^{[l]}(\bar{x}_1, x_2) - \delta L_{\Gamma}^{(1)}(\bar{x}_1, x_2)}$$

$$= \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}^{(1,2)}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}^{(1,2)}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{C_{2,l-2N_{\Gamma}}(x_1)} \\ \vdots \\ \overline{C_{2,l+2N_{\mathfrak{e}}-1}(x_1)} \end{bmatrix},$$

$$\sum_{k=0}^{l-1} \overline{\tilde{\Phi}_{2,k}^{(2,1)}(x_1) (\tilde{H}_k^{(2,1)})^{-1} \overline{L_{\Gamma}^{(2)}}(x_2) \tilde{C}_{1,k}^{(2,1)}(x_2) - \overline{L_{\Gamma}^{(2)}}(\bar{x}_1) K_{C_1}^{[l]}(\bar{x}_1, x_2) - \delta \overline{L_{\Gamma}^{(2)}}(\bar{x}_1, x_2)}$$

$$= \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} C_{1,l-2N_{\Gamma}}(x_2) \\ \vdots \\ C_{1,l+2N_{\mathfrak{e}}-1}(x_2) \end{bmatrix}.$$

Now, we compute the following spectral jets

$$\sum_{k=0}^{l-1} (\tilde{H}_k^{(1,2)})^{-1} \tilde{\Phi}_{1,k}^{(1,2)}(x_2) \overline{\mathcal{J}_{\tilde{C}_{2,k}^{(1,2)}}^{L_{\Gamma}^{(1)}}} - L_{\Gamma}^{(1)}(x_2) \overline{\mathcal{J}_{K_{C_2}^{[l]}}^{L_{\Gamma}^{(1)}}} - \overline{\mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{L_{\Gamma}^{(1)}}}(x_2)$$

$$= \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}^{(1,2)}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}^{(1,2)}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{L_{\Gamma}^{(1)}}} \end{bmatrix},$$

$$\sum_{k=0}^{l-1} \overline{\tilde{\Phi}_{2,k}^{(2,1)}(x_1) (\tilde{H}_k^{(2,1)})^{-1} \overline{\mathcal{J}_{\tilde{C}_{1,k}^{(2,1)}}^{L_{\Gamma}^{(2)}}} - \overline{L_{\Gamma}^{(2)}}(\bar{x}_1) \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(2)}}} - \overline{\mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{L_{\Gamma}^{(2)}}}(\bar{x}_1)$$

$$= \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{L_{\Gamma}^{(2)} C_{1,l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}}} \\ \vdots \\ \overline{\mathcal{J}_{L_{\Gamma}^{(2)} C_{1,l-1}}^{L_{\Gamma}^{(2)}}} \end{bmatrix}.$$

From (23) and (24) we deduce

$$\begin{aligned}
& \sum_{k=0}^{l-1} (\tilde{H}_k^{(1,2)})^{-1} \tilde{\Phi}_{1,k}^{(1,2)}(x_2) \overline{\langle L_{\mathfrak{e}}^{(1)} \xi^{(1)}, \tilde{\Phi}_{2,k}^{(1,2)} \rangle \mathcal{L}^{(1)}} - L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{C_2}^{[1]}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) - \mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) \\
&= \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}^{(1,2)}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}^{(1,2)}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\overline{\Gamma_{2,l}^{(1,2)}} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}}} \end{bmatrix}, \\
& \sum_{k=0}^{l-1} \overline{\tilde{\Phi}_{2,k}^{(2,1)}(x_1) (\tilde{H}_k^{(2,1)})^{-1} \langle L_{\mathfrak{e}}^{(2)} \xi^{(2)}, \tilde{\Phi}_{1,k}^{(2,1)} \rangle \mathcal{L}^{(2)}} - \overline{L_{\Gamma}^{(2)}}(\bar{x}_1) \mathcal{J}_{K_{C_1}^{[1]}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) - \mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) \\
&= \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{1,l-1}}^{\overline{L_{\Gamma}^{(2)}}}}(\bar{x}_2) \end{bmatrix}.
\end{aligned}$$

Recalling (3) we get

$$\begin{aligned}
& \left\langle \overline{L_{\mathfrak{e}}^{(2)}(z)(\xi^{(1)})_z}, \tilde{K}^{(1,2),[1]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} = L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{C_2}^{[1]}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) + \mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) \\
&+ \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}^{(1,2)}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}^{(1,2)}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\overline{\Gamma_{2,l}^{(1,2)}} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}}} \end{bmatrix}, \\
& \left\langle L_{\mathfrak{e}}^{(1)}(z)(\xi^{(2)})_z, \tilde{K}^{(2,1),[1]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} = \overline{L_{\Gamma}^{(2)}}(\bar{x}_1) \mathcal{J}_{K_{C_1}^{[1]}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) + \mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) \\
&+ \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}}} \\ \vdots \\ \overline{\mathcal{J}_{C_{1,l-1}}^{\overline{L_{\Gamma}^{(2)}}}}(\bar{x}_2) \end{bmatrix}.
\end{aligned}$$

Now, from (17) and (18), we deduce

$$\begin{aligned}
& \left\langle \overline{L_{\mathfrak{e}}^{(2)}(z)(\xi^{(1)})_z}, \tilde{K}^{(1,2),[1]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} = L_{\Gamma}^{(1)}(x_2) \left\langle \overline{(\xi^{(1)})_z}, K^{[1]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} \\
&+ \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}^{(1,2)}(z_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}^{(1,2)}(z_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\overline{\Gamma_{2,l}^{(1,2)}} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\langle \xi^{(1)}, \Phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} \\ \vdots \\ \overline{\langle \xi^{(1)}, \Phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)}} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
& \left\langle \overline{L_{\mathfrak{e}}^{(1)}(z)(\xi^{(2)})_z}, \tilde{K}^{(2),[l]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} = \overline{L_{\Gamma}^{(2)}(x_1)} \left\langle (\xi^{(2)})_z, K^{[l]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} \\
& + \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}(x_1)(\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}(x_1)(\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \langle \xi^{(2)}, \Phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)} \\ \vdots \\ \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)} \end{bmatrix}.
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
& L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{C_2}^{[l]}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) + \mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) - \overline{L_{\Gamma}^{(1)}(x_2)} \left\langle \overline{(\xi^{(1)})_z}, K^{[l]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} \\
& = - \left[\overline{\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}(x_2)(\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}}, \dots, \overline{\tilde{\Phi}_{1,l+2N_{\Gamma}-1}(x_2)(\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1}} \right] \\
& \quad \times \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} \\ \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \Phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)}} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
& \overline{L_{\Gamma}^{(2)}(x_1)} \mathcal{J}_{K_{C_1}^{[l]}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) + \mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) - \overline{L_{\Gamma}^{(2)}(x_1)} \left\langle (\xi^{(2)})_z, K^{[l]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} \\
& = - \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}(x_1)(\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}(x_1)(\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \\
& \quad \times \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} \\ \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}(x_2)}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \Phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)}} \end{bmatrix}.
\end{aligned}$$

We return to (17) and (18), and deduce, taking spectral jets, that

$$\begin{aligned}
(37) \quad & L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{[l]}}^{\overline{L_{\mathfrak{e}}^{(2)}}}(x_2) \\
& = - \left[\overline{\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}(x_2)(\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}}, \dots, \overline{\tilde{\Phi}_{1,l+2N_{\Gamma}-1}(x_2)(\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{e}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{\Phi_{2,l-2N_{\Gamma}}}^{\overline{L_{\mathfrak{e}}^{(2)}}}} \\ \vdots \\ \overline{\mathcal{J}_{\Phi_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\mathfrak{e}}^{(2)}}}} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
(38) \quad & \overline{L_{\Gamma}^{(2)}(x_1)} \mathcal{J}_{K_{[l]}}^{\overline{L_{\mathfrak{e}}^{(1)}}}(\bar{x}_1) \\
& = - \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}(x_1)(\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}(x_1)(\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{e}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{\Phi_{1,l-2N_{\Gamma}}}^{\overline{L_{\mathfrak{e}}^{(1)}}}} \\ \vdots \\ \overline{\mathcal{J}_{\Phi_{1,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\mathfrak{e}}^{(1)}}}} \end{bmatrix}.
\end{aligned}$$

Consequently,

$$\begin{aligned} & \left[L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{C_2}^{[1]}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) + \mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) - L_{\Gamma}^{(1)}(x_2) \left\langle \overline{(\xi^{(1)})_z}, K^{[1]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} \right] L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K^{[1]}}^{\overline{L_{\Gamma}^{(2)}}}(x_2) \\ &= - \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \\ & \quad \times \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{C}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(2)}}}} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & \left[L_{\Gamma}^{(2)}(x_1) \mathcal{J}_{K_{C_1}^{[2]}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) + \mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) - \overline{L_{\Gamma}^{(2)}}(x_1) \left\langle (\xi^{(2)})_z, K^{[2]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} \right] \overline{L_{\Gamma}^{(2)}}(x_1) \mathcal{J}_{K^{[2]}}^{\overline{L_{\Gamma}^{(1)}}}(\bar{x}_1) \\ &= - \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \\ & \quad \times \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{C}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}}} \end{bmatrix}. \end{aligned}$$

Hence, we conclude

$$\begin{aligned} & \left[\tilde{\Phi}_{1,l-2N_{\mathfrak{e}}}(x_2) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(1,2)})^{-1}, \dots, \tilde{\Phi}_{1,l+2N_{\Gamma}-1}(x_2) (\tilde{H}_{l+2N_{\Gamma}-1}^{(1,2)})^{-1} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \overline{\mathfrak{C}_{1,l}^{(1,2)}} \\ -\Gamma_{2,l}^{(1,2)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \\ &= - \left[L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K_{C_2}^{[1]}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) + \mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{\overline{L_{\Gamma}^{(1)}}}(x_2) - L_{\Gamma}^{(1)}(x_2) \left\langle \overline{(\xi^{(1)})_z}, K^{[1]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} \right] L_{\Gamma}^{(1)}(x_2) \mathcal{J}_{K^{[1]}}^{\overline{L_{\Gamma}^{(2)}}}(x_2) \\ & \quad \times \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(2)}}}} \end{bmatrix}^{-1}, \\ & \left[\overline{\tilde{\Phi}_{2,l-2N_{\mathfrak{e}}}^{(2,1)}(x_1) (\tilde{H}_{l-2N_{\mathfrak{e}}}^{(2,1)})^{-1}}, \dots, \overline{\tilde{\Phi}_{2,l+2N_{\Gamma}-1}^{(2,1)}(x_1) (\tilde{H}_{l+2N_{\Gamma}-1}^{(2,1)})^{-1}} \right] \begin{bmatrix} 0_{2N_{\mathfrak{e}} \times 2N_{\Gamma}} & \mathfrak{C}_{1,l}^{(2,1)} \\ -\Gamma_{1,l}^{(2,1)} & 0_{2N_{\Gamma} \times 2N_{\mathfrak{e}}} \end{bmatrix} \\ &= - \left[\overline{L_{\Gamma}^{(2)}(x_1) \mathcal{J}_{K_{C_1}^{[2]}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) + \mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{\overline{L_{\Gamma}^{(2)}}}(\bar{x}_1) - \overline{L_{\Gamma}^{(2)}}(x_1) \left\langle (\xi^{(2)})_z, K^{[2]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} \right] \overline{L_{\Gamma}^{(2)}}(x_1) \mathcal{J}_{K^{[2]}}^{\overline{L_{\Gamma}^{(1)}}}(\bar{x}_1) \\ & \quad \times \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}}}^{\overline{L_{\Gamma}^{(1)}}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(2)}}} - \langle \xi^{(2)}, \phi_{1,l+2N_{\mathfrak{e}}-1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l+2N_{\mathfrak{e}}-1}}^{\overline{L_{\Gamma}^{(1)}}}} \end{bmatrix}^{-1}. \end{aligned}$$

Recalling (9) we get

$$\begin{aligned} \tilde{\Phi}_{1,l}^{(1,2)}(x_2) &= \left[L_{\Gamma}^{(1)}(x_2) \overline{\mathcal{J}_{K_{C_2}^{[l]}}^{L_{\Gamma}^{(1)}}(x_2)} + \overline{\mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{L_{\Gamma}^{(1)}}(x_2)} - \overline{L_{\Gamma}^{(1)}(x_2)} \left\langle \overline{(\xi^{(1)})_z}, K^{[l]}(\bar{z}, x_2) \right\rangle \overline{\mathcal{L}^{(1)}} \right] L_{\Gamma}^{(1)}(x_2) \overline{\mathcal{J}_{K_{C_2}^{[l]}}^{L_{\Gamma}^{(2)}}(x_2)} \\ &\quad \times \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(1)}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\Gamma}-1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(2)}}} \end{bmatrix}^{-1} \begin{bmatrix} \overline{H_{l-2N_{\Gamma}}} \\ \overline{L_{\Gamma,(-1)^{l_{N_{\Gamma}}}}^{(1)}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\ \tilde{\Phi}_{1,l}^{(2,1)}(x_1) &= \left[\overline{L_{\Gamma}^{(2)}(x_1)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(2)}}(x_1)} + \overline{\mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{L_{\Gamma}^{(2)}}(x_1)} - \overline{L_{\Gamma}^{(2)}(x_1)} \left\langle (\xi^{(2)})_z, K^{[l]}(\bar{x}_1, z) \right\rangle \mathcal{L}^{(2)} \right] \overline{L_{\Gamma}^{(2)}(x_1)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(1)}}(x_1)} \\ &\quad \times \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(2)}}(x_2)} - \langle \xi^{(2)}, \phi_{1,l+2N_{\Gamma}-1} \rangle \mathcal{L}^{(2)} & \overline{\mathcal{J}_{\phi_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(1)}}} \end{bmatrix}^{-1} \begin{bmatrix} \overline{H_{l-2N_{\Gamma}}} \\ \overline{L_{\Gamma,(-1)^{l_{N_{\Gamma}}}}^{(2)}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, have proven the following last quasideterminantal expressions

$$\begin{aligned} \tilde{\Phi}_{2,l}^{(2,1)}(z) &= -\Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}}} & \overline{H_{l-2N_{\Gamma}}} \\ \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}+1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(1)}}} & 0 \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l+2N_{\Gamma}-1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(1)}}} & 0 \\ \overline{L_{\Gamma}^{(2)}(z)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(2)}}(z)} + \overline{\mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{L_{\Gamma}^{(2)}}(z)} - \overline{L_{\Gamma}^{(2)}(z)} \left\langle (\xi^{(2)})_w, K^{[l]}(\bar{z}, w) \right\rangle \mathcal{L}^{(2)} & \overline{L_{\Gamma}^{(2)}(z)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(1)}}(z)} & 0 \end{bmatrix}, \\ \tilde{\Phi}_{1,l}^{(1,2)}(z) &= -\Theta_* \begin{bmatrix} \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(1)}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_{\Gamma}}}^{L_{\Gamma}^{(2)}}} & \overline{H_{l-2N_{\Gamma}}} \\ \overline{\mathcal{J}_{C_{2,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(1)}} - \langle \xi^{(1)}, \phi_{2,l-2N_{\Gamma}+1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(2)}}} & 0 \\ \vdots & \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(1)}} - \langle \xi^{(1)}, \phi_{2,l+2N_{\Gamma}-1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(2)}}} & 0 \\ \overline{L_{\Gamma}^{(1)}(z)} \overline{\mathcal{J}_{K_{C_2}^{[l]}}^{L_{\Gamma}^{(1)}}(z)} + \overline{\mathcal{J}_{\delta L_{\Gamma}^{(1)}}^{L_{\Gamma}^{(1)}}(z)} - \overline{L_{\Gamma}^{(1)}(z)} \left\langle \overline{(\xi^{(1)})_w}, K^{[l]}(\bar{w}, z) \right\rangle \overline{\mathcal{L}^{(1)}} & \overline{L_{\Gamma}^{(1)}(z)} \overline{\mathcal{J}_{K_{C_2}^{[l]}}^{L_{\Gamma}^{(2)}}(z)} & 0 \end{bmatrix}. \end{aligned}$$

These formulas can be expressed in terms of determinants

$$\tilde{\Phi}_{2,l}^{(2,1)}(z) = -\frac{H_{l-2N_{\Gamma}}}{\overline{L_{\Gamma,(-1)^{l_{N_{\Gamma}}}}^{(2)}}} \frac{1}{\tilde{\tau}_l^{(2,1)}} \begin{vmatrix} \overline{\mathcal{J}_{C_{1,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l-2N_{\Gamma}+1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l-2N_{\Gamma}+1}}^{L_{\Gamma}^{(1)}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(2)}} - \langle \xi^{(2)}, \phi_{1,l+2N_{\Gamma}-1} \rangle \mathcal{L}^{(2)}} & \overline{\mathcal{J}_{\phi_{1,l+2N_{\Gamma}-1}}^{L_{\Gamma}^{(1)}}} \\ \overline{L_{\Gamma}^{(2)}(z)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(2)}}(z)} + \overline{\mathcal{J}_{\delta L_{\Gamma}^{(2)}}^{L_{\Gamma}^{(2)}}(z)} - \overline{L_{\Gamma}^{(2)}(z)} \left\langle (\xi^{(2)})_w, K^{[l]}(\bar{z}, w) \right\rangle \mathcal{L}^{(2)} & \overline{L_{\Gamma}^{(2)}(z)} \overline{\mathcal{J}_{K_{C_1}^{[l]}}^{L_{\Gamma}^{(1)}}(z)} \end{vmatrix},$$

$$\tilde{\Phi}_{1,l}^{(1,2)}(z) = -\frac{H_{l-2N_\Gamma}}{L_{\Gamma,(-1)^{N_\Gamma}} \tilde{\tau}_l^{(1,2)}} \begin{vmatrix} \overline{\mathcal{J}_{C_{2,l-2N_\Gamma+1}}^{L_\Gamma^{(1)}} - \langle \xi^{(1)}, \phi_{2,l-2N_\Gamma+1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l-2N_\Gamma+1}}^{L_\epsilon^{(2)}}} \\ \vdots & \vdots \\ \overline{\mathcal{J}_{C_{2,l+2N_\epsilon-1}}^{L_\Gamma^{(1)}} - \langle \xi^{(1)}, \phi_{2,l+2N_\epsilon-1} \rangle \mathcal{L}^{(1)}} & \overline{\mathcal{J}_{\phi_{2,l+2N_\epsilon-1}}^{L_\epsilon^{(2)}}} \\ L_\Gamma^{(1)}(z) \mathcal{J}_{K_{C_2}^{[l]}}^{L_\Gamma^{(1)}}(z) + \mathcal{J}_{\delta L_\Gamma^{(1)}}^{L_\Gamma^{(1)}}(z) - L_\Gamma^{(1)}(z) \langle \overline{(\xi^{(1)})_w}, K^{[l]}(\bar{w}, z) \rangle \overline{\mathcal{L}^{(1)}} & L_\Gamma^{(1)}(z) \mathcal{J}_{K^{[l]}}^{L_\epsilon^{(2)}}(z) \end{vmatrix}.$$

But, we have the following linear expressions for the last rows of these matrices

$$\begin{aligned} & \left[\overline{L_\Gamma^{(2)}(z) \mathcal{J}_{K_{C_1}^{[l]}}^{L_\Gamma^{(2)}}(\bar{z})} + \overline{\mathcal{J}_{\delta L_\Gamma^{(2)}}^{L_\Gamma^{(2)}}(\bar{z})} - \overline{L_\Gamma^{(2)}(z) \langle (\xi^{(2)})_w, K^{[l]}(\bar{z}, w) \rangle \mathcal{L}^{(2)}} \right] \overline{L_\Gamma^{(2)}(z) \mathcal{J}_{K^{[l]}}^{L_\epsilon^{(1)}}(\bar{z})} \\ &= \left[\overline{L_\Gamma^{(2)}(z) \mathcal{J}_{K_{C_1}^{[l-2N_\Gamma+1]}}^{L_\Gamma^{(2)}}(\bar{z})} + \overline{\mathcal{J}_{\delta L_\Gamma^{(2)}}^{L_\Gamma^{(2)}}(\bar{z})} - \overline{L_\Gamma^{(2)}(z) \langle (\xi^{(2)})_w, K^{[l-2N_\Gamma+1]}(\bar{z}, w) \rangle \mathcal{L}^{(2)}} \right] \overline{L_\Gamma^{(2)}(z) \mathcal{J}_{K^{[l-2N_\Gamma+1]}}^{L_\epsilon^{(1)}}(\bar{z})} \\ &+ \overline{L_\Gamma^{(2)}(z)} \sum_{k=l-2N_\Gamma+1}^{l-1} \overline{\phi_{2,k}(z)} (H_k)^{-1} \left[\overline{\mathcal{J}_{C_{1,k}}^{L_\Gamma^{(2)}} - \langle \xi^{(2)}, \phi_{1,k} \rangle \mathcal{L}^{(2)}} \right] \overline{\mathcal{J}_{\phi_{1,k}}^{L_\epsilon^{(1)}}}, \\ & \left[L_\Gamma^{(1)}(z) \mathcal{J}_{K_{C_2}^{[l]}}^{L_\Gamma^{(1)}}(z) + \mathcal{J}_{\delta L_\Gamma^{(1)}}^{L_\Gamma^{(1)}}(z) - L_\Gamma^{(1)}(z) \langle \overline{(\xi^{(1)})_w}, K^{[l]}(\bar{w}, z) \rangle \overline{\mathcal{L}^{(1)}} \right] L_\Gamma^{(1)}(z) \mathcal{J}_{K^{[l]}}^{L_\epsilon^{(2)}}(z) \\ &= \left[L_\Gamma^{(1)}(z) \mathcal{J}_{K_{C_2}^{[l-2N_\Gamma+1]}}^{L_\Gamma^{(1)}}(z) + \mathcal{J}_{\delta L_\Gamma^{(1)}}^{L_\Gamma^{(1)}}(z) - L_\Gamma^{(1)}(z) \langle \overline{(\xi^{(1)})_w}, K^{[l-2N_\Gamma+1]}(\bar{w}, z) \rangle \overline{\mathcal{L}^{(1)}} \right] L_\Gamma^{(1)}(z) \mathcal{J}_{K^{[l-2N_\Gamma+1]}}^{L_\epsilon^{(2)}}(z) \\ &+ L_\Gamma^{(1)}(z) \sum_{k=l-2N_\Gamma+1}^{l-1} \phi_{1,k}(z) (H_k)^{-1} \left[\overline{\mathcal{J}_{C_{2,k}}^{L_\Gamma^{(1)}} - \langle \xi^{(1)}, \phi_{2,k} \rangle \mathcal{L}^{(1)}} \right] \overline{\mathcal{J}_{\phi_{2,k}}^{L_\epsilon^{(2)}}}, \end{aligned}$$

in where we see that the last term in the RHS is a linear combination of the rows present in the matrix and consequently, can be disregarded in the computation of the determinant and we find (27) and (30). \square

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